# Section 2 Proof and Logic

Paper 2 of the TMUA tests the candidate's ability to think mathematically, and their ability to understand, and construct, mathematical arguments in a variety of contexts. Questions may draw on any mathematical knowledge from Section 1 (the eight topic areas MM1 to MM8 plus the seven 'GCSE' topics M1 to M7) . However the majority of questions will be phrased in terms of logical arguments or proofs.

The specification introduces a number of key terms and their mathematical definitions, which may be unfamiliar to A Level students. It is crucial to understand these terms, and be able to use them to analyse or construct proofs.

The worksheet I have developed introduces all these concepts and provides examples and practice questions so that students can become familiar with the terminology and how it is used.

There is also a PDF, 'Notes on Logic and Proof', published by Cambridge Assessment Admissions Testing. This also covers the key material but it is 65 pages long and with no solutions, so many students do not find it quite as helpful to start with. But it is certainly helpful to read through it once you have grasped the basics.

# Specification for TMUA

**The Logic of Arguments**

Arg 1 Understand and be able to use mathematical logic in simple situations:

- The terms true and false;
- The terms and, or (meaning inclusive or), not;
- Statements of the form: if **A** then **B**<br>**A** if **B**<br>**A** only if **B**<br>**A** if and only if **B**
- The converse of a statement;
- The contrapositive of a statement;
- The relationship between the truth of a statement and its converse and its contrapositive.

**Note:** candidates will **not** be expected to recognise or use symbolic notation for any of these terms, nor will they be expected to complete formal truth tables.

Arg2 Understand and use the terms **necessary** and **sufficient**.

Arg3 Understand and use the terms **for all**, **for some** (meaning for at least one), and **there exists**.

Arg4 Be able to negate statements that use any of the above terms.

**Mathematical Proof**

Prf1 Follow a proof of the following types, and in simple cases know how to construct such a proof:

- Direct deductive proof ('Since A, therefore B, … , therefore Z, which is what we wanted to prove.');
- Proof by cases (for example, by considering even and odd cases separately);
- Proof by contradiction;
- Disproof by counterexample.

Prf2 Deduce implications from given statements.

Prf3 Make conjectures based on small cases, and then justify these conjectures.

Prf4 Rearrange a sequence of statements into the correct order to give a proof for a statement.

Prf5 Problems requiring a sophisticated chain of reasoning to solve.

**Identifying Errors in Proofs**

Err1 Identifying errors in purported proofs.

Err2 Be aware of common mathematical errors in purported proofs; for example, claiming 'if $ab = ac$, then $b = c$' or assuming 'if $\sin A = \sin B$, then $A = B$' neither of which are valid deductions.

## The Logic of Arguments

### Statements

A statement is a sentence which is definitely true or definitely false - it can never be both.

In the TMUA we may need to consider the following types of statement:

- statements that are obviously true or obviously false

    eg "5 is a prime number" or "5 is greater than 6"

- statements that need some work to decide if they are true or not

    eg "4583641 is a prime number"

- statements that may be true for certain values

    eg "the square of an integer $x$ is odd" would be a true statement for all odd values of $x$

### Conditional Statements

We can combine statements together to form **conditional statements**, sometimes known as **if/then statements**, where '**if**' gives the hypothesis, and '**then**' gives the conclusion.

**If** an integer is odd, **then** the square of the integer is odd (this is true)

the hypothesis is "an integer is odd" / the conclusion is "the square of the integer is odd"

**If** an integer is prime, **then** it is an odd number (this is false)

the hypothesis is "an integer is prime" / the conclusion is "it is an odd number"

### Implication of 'If' statements

The most important thing to remember is that conditional statements tell us what happens if the hypothesis is *true*, but they don't tell us anything about what happens if the hypothesis is *false*.

Example: **If** it is raining **then** I will wear a coat

This tells me that if it rains, I will wear a coat, but if it is not raining I can not draw any conclusion - I might or might not wear a coat in this situation.

*Be very careful not to infer too much information from a conditional statement.*

Logic of Arguments

1a. Consider the following statement: "If it is my birthday, I will eat some cake"

What conclusion can I draw from each of the following statements: birthday $\Rightarrow$ cake

i) It is my birthday — I will eat cake

ii) It is not my birthday — No conclusion - might eat cake / might not

iii) I eat some cake — No conclusion - might eat cake on another day so could be my birthday but might not be

iv) I do not eat some cake — It is not my birthday (because if it was my birthday I would eat cake)

1b. Consider the following statement: "If it rains the ground will get wet"

What conclusion can I draw from each of the following statements: rain $\Rightarrow$ wet

i) The ground is wet — No conclusion - might be wet from rain or from a sprinkler

ii) The ground is not wet — It has not been raining

iii) It is raining — The ground will get wet

iv) It is not raining — No conclusion - ground might be wet or dry

1c. Consider the following statement: "If I am in Paris, then I am in France"

What conclusion can I draw from each of the following statements:

i) I am in Paris — I am in France

ii) I am in France — no conclusion - might be in Paris or in Lille...

iii) I am in London — = I am not in France $\Rightarrow$ I am not in Paris

iv) I am at the Eiffel Tower — $\Rightarrow$ I am in Paris $\Rightarrow$ I am in France

1d. Consider the following statement: "If a shape is a square, then it is a quadrilateral"

What conclusion can I draw from each of the following statements:

i) The shape is a square — it is a quadrilateral

ii) The shape is a quadrilateral — no conclusion - might be a square or a parallelogram...

iii) The shape is not a quadrilateral — it is not a square

iv) The shape is a rhombus — it is not a square

There are different ways to write a conditional statement which are all logically equivalent, as follows:

(*) **If** it is raining **then** I will wear a coat

a) I will wear a coat **if** it is raining

b) It is raining **only if** I wear a coat

c) Wearing a coat is **necessary** when it is raining.

d) Rain is **sufficient** for me to wear a coat

(*) The original statement has hypothesis "It is raining" and conclusion "I will wear a coat"
Let's use the shorthand **If** Rain **then** Coat

Look at the equivalent cases a) to d) above, noting the relative position of the hypothesis and conclusion, and how the meaning of each case stays the same.

Note that I can not draw any conclusion from the hypothesis "If I wear a coat" as the original statement doesn't tell me anything about this situation.

Let's consider each case in turn:

a) In this case the whole '**if hypothesis**' is switched to the end of the statement and the **conclusion** is given at the start, which gives us Coat **if** Rain
This is obviously a logically equivalent statement.

b) In this case the word **if** in front of the hypothesis in (*) has been removed and replaced with the words **only if** in front of of the conclusion, which gives us Rain **only if** Coat
So the only way it can be raining is if I am wearing a coat.

(This is often the case that people struggle with the most. It does not tell me anything about what is true *if I am wearing a coat.* Instead it is setting out what must be true in order for it to be raining - ie *I must be wearing a coat*. Or the only way it can be raining is if I am wearing a coat.)

c) The word **if** from case a) is replaced with **is necessary**, giving Coat **necessary** for Rain
In other words wearing a coat is necessary when it is raining

d) The words **only if** from case b) are replaced with **is sufficient** giving Rain **sufficient** for Coat
So if it is raining, that is a sufficient reason for me to wear a coat.

These cases illustrate that you can swap if / only if to reverse the hypothesis and conclusion in a conditional statement, for example

A if B = B only if A (A=Coat / B=Rain)

Similarly with necessary / sufficient

A is necessary for B = B is sufficient for A

**Necessary** If A is necessary for B, then B can't happen without A, so we have '**if B then A**'.

**Sufficient** If B is sufficient for A, then A must happen if B happens, so we have '**if B then A**'.

It is often useful to rewrite statements in the form "if *hypothesis* then *conclusion*", so it is easier to compare which ones have the same meaning.

2. Rewrite the following true statements in the form **If… Then …**

a) The ground gets wet when it rains

If it rains, then the ground gets wet

b) All mammals have hair

If it is a mammal, then it has hair

c) I can ride the roller coaster only if I am tall enough

If I ride the roller coaster, then I am tall enough

d) A fruit is yellow if it is a banana

If its a banana, then it is yellow

e) I am in Paris only if I am in France

If I'm in Paris, then I'm in France

f) Having a passport is necessary for travelling abroad.

If I travel abroad, then I have a passport.

g) Coming first is sufficient for winning a medal.

If I come first, then I win a medal.

3. Rewrite the following true mathematical statements in the form **If… Then …**

a) Any rectangle is a quadrilateral — If its a rectangle, then its a quadrilateral

b) All triangles have 3 sides — If its a triangle, then it has 3 sides

c) The number 2 is the only even prime number — If its an even prime number, then its 2
If its 2, then its an even prime number

d) $x > 10$ if $x > 100$ — If $x > 100$, then $x > 10$

e) $k^2 < 1$ only if $k < 1$ — If $k^2 < 1$, then $k < 1$ (converse is not true)

f) $p^2 < p$ is sufficient for $p < 1$ — If $p^2 < p$, then $p < 1$ (converse is not true)

g) $k$ being an even number is necessary for $k$ to be divisible by 4

If $k$ is divisible by 4, then $k$ is an even number

4. Are the following statements true or false? Find counterexample

| | | | | |
|---|---|---|---|---|
| a) | $x > 5$ | if $\Leftarrow$ | $x > 10$ | True. If $x > 10$ then $x > 5$ |
| b) | $x < 8$ | only if $\Rightarrow$ | $x < 3$ | False eg $x = 5$ (Converse is true) |
| c) | x is even | if and only if $\Leftrightarrow$ | $(x + 1)$ is odd | True |
| d) | $ab = ac$ | ✓ if and only if ✗ $\Leftarrow$ $\Rightarrow$ | $b = c$ | 'only if' is False $a = 0$ \| 'If' is True $b = c \Rightarrow ab = ac$ so False |
| e) | $a^2 < a$ | if $\Leftarrow$ | $a < 1$ | False $a = -2$ |
| f) | $a^2 < a$ | only if $\Rightarrow$ | $a < 1$ | True (This is converse of (e)) |
| g) | $a^2 < a$ $a(a-1) < 0$ $0 < a < 1$ | if and only if $\Leftarrow$ $\Rightarrow$ | $-1 < a < 1$ | 'only if' is True \| 'If' is False eg. $a = -\frac{1}{2}$ so False |
| h) | an even number is prime | if and only if $\Leftarrow$ $\Rightarrow$ | it is 2 | True |
| i) | an odd number is prime | if and only if $\Leftarrow$ $\Rightarrow$ | it is 3 | 'only if' is False eg 7 \| 'If' is True 3 is odd prime so False |
| j) | a triangle is equilateral | if and only if $\Leftarrow$ $\Rightarrow$ | all it angles are 60° | True |
| k) | a triangle is isosceles | if $\Leftarrow$ | it is equilateral | True (converse is False) |

5. Complete the statements with one of the following:

‘necessary’ ‘sufficient’ ‘necessary and sufficient’ ‘not necessary and not sufficient’

$\Leftarrow$ N  $\Rightarrow$ S  $\Leftrightarrow$ N & S  $\not\Leftrightarrow$ not N & not S

a) having 2 legs is a ___N___ condition for being a bird (bird $\Rightarrow$ 2 legs)

b) being a robin is a ___S___ condition for being a bird (robin $\Rightarrow$ bird)

c) being a bird is a ___N & S___ condition for having feathers
(only birds have feathers and all birds have feathers)

d) having feathers is a ___not N / not S___ condition for being able to fly
(penguins can't fly / bats don't have feathers)

e) being an odd number is a ___N___ condition for being a prime number greater than 10
prime > 10 $\Rightarrow$ odd

f) being greater then 20 is a ___S___ condition for being a greater than 10
$x > 20 \Rightarrow x > 10$
$x > 10 \not\Rightarrow x > 20$ eg $x = 12$

g) being a stationary point is a ___N___ condition for being a maximum point
max $\Rightarrow$ stationary
stationary $\not\Rightarrow$ maximum

h) $x^2 = 1$ is ___N___ for $x = 1$
$x = 1 \Rightarrow x^2 = 1$
$x^2 = 1 \not\Rightarrow x = 1$ eg $x = -1$

i) $x^2 < 5$ is ___S___ for $x^2 < 10$
$x^2 < 5 \Rightarrow x^2 < 10$
$x^2 < 10 \not\Rightarrow x^2 < 5$ eg $x^2 = 6$

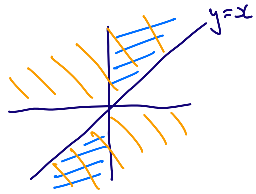
j) $x^2 < 1$ is ___N & S___ for $-1 < x < 1$

k) $ab < ac$ is ___not N & not S___ for $b < c$
not N eg $a = 0,\ a < 0$
not S eg $a = -1,\ a < 0$

l) $\int_{-1}^{1} f(x) = 0$ is ___N___ for $f(x)$ to be an odd function
odd $f(-x) = -f(x)$ $\int_{-1}^{1} f(x) = 0$ $\Leftarrow$ true
$\int_{-1}^{1} f(x) = 0$  not odd $\Rightarrow$ false

m) $\frac{x}{y} < 1$ is ___N___ for $\frac{y}{x} > 1$
$x = 1 \quad y = -2$
$\frac{x}{y} = -\frac{1}{2} \quad \frac{y}{x} = -2 \Rightarrow$ false

$\frac{x-y}{y} < 0$
$y < 0 \quad x > y$
$y > 0 \quad x < y$




$\frac{y-x}{x} > 0$
$x > 0 \quad y > x$ $x, y$ both positive
$x < 0 \quad y < x$ $x, y$ both negative

How to answer: Is P sufficient for Q? In other words do we have *If P then Q*?

Look for a counter example - an instance where P is true but Q is not true

Or look for proof that P implies Q

How to answer: Is P necessary for Q? In other words do we have *If Q then P*?

Look for a counter example - an instance where Q is true but not P

Or look for proof that Q implies P

6. Consider the following pairs of statements.
For each pair, determine if P is sufficient and / or necessary for Q

a) P Having a son
Q Being a parent

$\Downarrow$ ✓ $\Uparrow$ x could have daughter

P is sufficient for Q

b) P Being on the third floor
Q Climbing the stairs to the third floor

$\Uparrow$ ✓ $\Downarrow$ x could take lift to 3rd floor

P is necessary for Q

c) P Being tall
Q Being successful

$\Updownarrow$ x

P is not sufficient or necessary for Q

d) P An integer being an even prime number
Q An integer being the number 2

$\Updownarrow$

P is sufficient and necessary for Q

e) P A function being a cubic polynomial
Q A function having at least one real root

$\Downarrow$ ✓ $\Uparrow$ x could be any function that crosses x-axis

P is sufficient for Q

f) P An integer being divisible by 5
Q An integer ending in a 5

$\Uparrow$ ✓ $\Downarrow$ x eg 50

P is necessary for Q

g) P $\sin x = \sin y$
Q $x = y$

$\Uparrow$ ✓ $\Downarrow$ x eg $\sin 30 = \sin 150$

P is necessary for Q

h) P $|x| < 25$
Q $x \neq 25$

$\Downarrow$ ✓ $\Uparrow$ x eg $x = 30$

P is sufficient for Q

The following list of statements are all logically equivalent to each other. In particular look at the final three examples which introduce some **negation** or ‘**not**’ statements.

> **If** an animal is a zebra, **then** it has stripes
>
> **All** zebras have stripes
>
> **Any** zebra has stripes
>
> Being a zebra **implies** having stripes
>
> An animal is a zebra **only if** it has stripes
>
> An animal has stripes **if** it is a zebra
>
> Being a zebra is a **sufficient** condition for an animal to have stripes
>
> Having stripes is **necessary** for an animal to be a zebra
>
> An animal with **no** stripes is **not** a zebra
>
> Having **no** stripes is **sufficient** for **not** being a zebra
>
> **Not** being a zebra is **necessary** for **not** having stripes

**Converse and Contrapositive**

The **converse** of a conditional statement swaps the hypothesis and conclusion but is *not always true*.

Consider the original statement: (*) **If** it is raining **then** I will wear a coat

The converse of this statement is: If I wear a coat, then it is raining

> but this may not be true as I might wear a coat when it is cold but not raining.

However the original statement does tell me something when I am **not** wearing a coat. In this case it can’t be raining, because if it was I would wear a coat which is a contradiction. Therefore I can say:

> If I am **not** wearing a coat, then it is **not** raining.

This is called the ***contrapositive*** and is always logically equivalent to the original statement. It is obtained by negating both hypothesis and conclusion and swapping them.

Other equivalent ways of expressing this contrapositive include:

> **Not** wearing a coat is **sufficient** for it **not** to be raining
>
> **Not** raining is **necessary** for me **not** to wear a coat
>
> I will **not** wear a coat **only if** it is **not** raining
>
> It is **not** raining **if** I am **not** wearing a coat

7. Write the contrapositive of the following statements (be careful to identify the hypothesis and conclusion before swapping them):

a) **If** I have enough money, I will go on holiday

If I don't go on holiday then I don't have enough money

b) **If** you do not study, you will not do well in your exams

If you do well in your exams, then you did study

c) Ben will not go to school **only if** he is sick

(= no school $\Rightarrow$ sick) If Ben is not sick, he will go to school

d) I will wear a hat **if** it is sunny

(= If sunny, then hat) If I don't wear a hat, then it's not sunny

e) Renting a flat is **sufficient** to have somewhere to live

(= If flat, then live) If I don't have somewhere to live, then I am not renting a flat.

f) Practising the piano is **necessary** to pass a piano exam

(= If pass, then practise) If I don't practise the piano, then I can't pass the piano exam

8. Write the contrapositive of the following mathematical statements:

a) **If** a shape has 4 sides, it is a quadrilateral

If not a quadrilateral, then doesn't have 4 sides

b) **If** an integer is not equal to 2, then it is not an even prime

If even prime, then equal to 2

c) A number is even **only if** the square of the number is even

(= If even, then square is even) If square is not even, then number not even OR If square is odd, then number is odd

d) $f(a) > 0$ **if** $a > 0$

(= If $a > 0$ then $f(a) > 0$) If $f(a) \leq 0$ then $a \leq 0$

e) $a^2 < a$ is a **sufficient** condition for $a < 1$

(= If $a^2 < a$ then $a < 1$) If $a \geq 1$ then $a^2 \geq a$

f) $a^2 > 9$ is a **necessary** condition for $a > 3$

(= If $a > 3$ then $a^2 > 9$) If $a^2 \leq 9$ then $a \leq 3$

1. $a$ and $b$ are real numbers and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

Let $P = a<b$
$Q = f(a)<f(b)$

A **If** $a<b$ **then** $f(a)<f(b)$ — If P then Q
B $a<b$ **only if** $f(a)<f(b)$ — P only if Q = If P then Q
C $f(a)<f(b)$ is **sufficient** for $a<b$ — Q is sufficient for P = If Q then P
D $f(a)<f(b)$ is **necessary** for $a<b$ — Q is necessary for P = If P then Q

2. $a$ is a real number and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $\overset{P}{a>0}$ **then** $\overset{Q}{f(a)>0}$ — If P then Q
B $a>0$ **only if** $f(a)>0$ — P only if Q = If P then Q
C $a>0$ is **sufficient** for $f(a)>0$ — P sufficient for Q = If P then Q
D $a>0$ is **necessary** for $f(a)>0$ — P necessary for Q = If Q then P

3. $a$ is a real number and $f$ is a function.
Given that exactly one of the following statements is true, which one is it?

A **If** $\overset{P}{f(a)>0}$ **then** $\overset{Q}{|a|<1}$ — If P then Q
B $f(a)>0$ **if** $|a|<1$ — P if Q = if Q then P
C $|a|<1$ **only if** $f(a)>0$ — Q only if P = if Q then P
D $|a|<1$ is **sufficient** for $f(a)>0$ — Q sufficient for P = if Q then P

P: even prime Q: $x=2$

4. Given that exactly one of the following statements is true, which one is it?

A $x$ is not an even prime **only if** $x=2$ — not P only if Q = If not P then Q = If not Q then P
B **if** $x$ is an even prime, **then** $x \neq 2$ — if P then not Q
C $x \neq 2$ is **sufficient** for $x$ to be an even prime — not Q sufficient for P = if not Q then P
D $x \neq 2$ is **necessary** for $x$ to be an even prime — not Q necessary for P = if P then not Q
E $x=2$ **if and only if** $x$ is not an even prime — if not P then Q; if Q then not P
F $x$ is not an even prime **only if** $x \neq 2$ — not P only if not Q = if not P then not Q = if Q then P

5. f is a function and *a is* a real number. P: $a \le 0$  Q: $f(a) \le 0$

Given that exactly one of the following statements is true, which one is it?

A  $a \le 0$ **only if** $f(a) \le 0$  P only if Q : if P then Q

B  $f(a) > 0$ **if** $a > 0$  not Q if not P : if not P then not Q : if Q then P

C  $f(a) > 0$ is **sufficient** for $a > 0$  not Q sufficient for not P : if not Q then not P : if P then Q

D  $f(a) \le 0$ is **necessary** for $a \le 0$  Q necessary for P : if P then Q

E  **If** $f(a) > 0$ **then** $a > 0$  if not Q then not P : if P then Q

F  $a > 0$ **if** $f(a) > 0$  not P if not Q : if not Q then not P : if P then Q

6. f is a function and *a, b* are real numbers.  P ⇒ Q

Given that exactly one of the following statements is true, which one is it?  not Q ⇒ not P

A  $f(a) \ge f(b)$ if and only if $a \ge b$  If $f(a) \ge f(b)$ then $a \ge b$ / If $a \ge b$ then $f(a) \ge f(b)$

B  $f(a) \ge f(b)$ only if $a < b$  If $f(a) \ge f(b)$ then $a < b$  contrapositive

C  $f(a) < f(b)$ if $a \ge b$  If $a \ge b$ then $f(a) < f(b)$

D  $a \ge b$ if $f(a) \ge f(b)$  If $f(a) \ge f(b)$ then $a \ge b$

E  $a < b$ only if $f(a) \ge f(b)$  If $a < b$ then $f(a) \ge f(b)$  contrapositive

F  $a < b$ only if $f(a) < f(b)$  If $a < b$ then $f(a) < f(b)$

equivalent

A, D equivalent

B, C  D, F contrapositive

Let $x$ be a real number.

7. Which **one** of the statements below is a **necessary** condition for **all** of the other four statements? D

8. Which **one** of the statements below is a **sufficient** condition for **all** of the other four statements? B

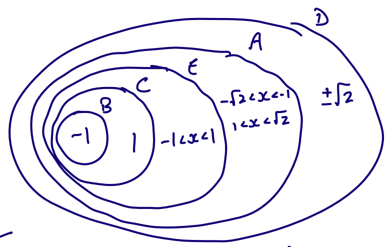
A  $x^2 < 2$  ⇒ D

B  $x = -1$  ⇒ A, C, D, E

C  $x^2 = 1$  ⇒ A, D, E

D  $x^2 \le 2$  ⇒ x

E  $x^2 \le 1$  ⇒ A, D




or

Order the conditions within a Venn diagram

If something is in the middle circle (B) then it is inside all the other circles

∴ B sufficient for C, E, A, D

Let $x$ be a real number.

9. Which **one** of the statements below is a **sufficient** condition for **all** of the other four statements? C

10. Which **one** of the statements below is a **necessary** condition for **all** of the other four statements? B

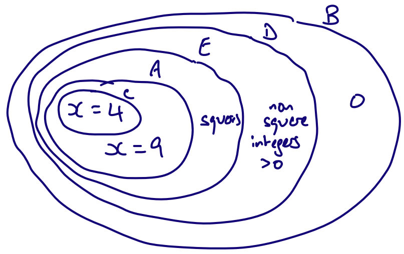
A $\quad x = 4$ or $x = 9$ $\quad \Rightarrow B, D, E$

B $\quad x \geq 0$ $\quad \Rightarrow \times$

C $\quad x = 4$ $\quad \Rightarrow A, B, D, E$

D $\quad x$ is a positive integer $\quad \Rightarrow B$

E $\quad x$ is a positive square number $\quad \Rightarrow B, D$




Let $x$ be a real number. Which **one** of the statements below…

11. … is a **sufficient** condition for **exactly 3** of the other four statements? B

12. … is a **necessary** condition for **exactly 3** of the other four statements? E

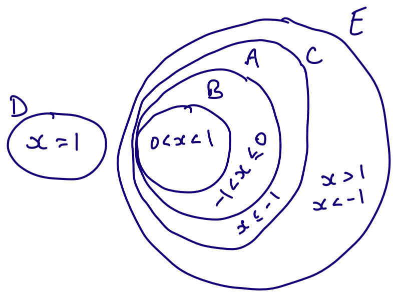
A $\quad x^2 < 1$ $\quad \Rightarrow C, E$

B $\quad 0 < x < 1$ $\quad \Rightarrow A, C, E$

C $\quad x < 1$ $\quad \Rightarrow E$

D $\quad x = 1$ $\quad \Rightarrow \times$

E $\quad x \neq 1$ $\quad \Rightarrow \times$




13. Consider the four options below about a particular statement (*):

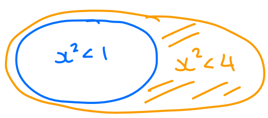
A $\quad$ The statement (*) is true if $x^2 < 1$ $\quad$ If $x^2 < 1$ then statement is true

B $\quad$ The statement (*) is true if and only if $x^2 < 1$ $\Rightarrow$ B + A true $\quad \times$

C $\quad$ The statement (*) is true if $x^2 < 4$ $\quad$ If $x^2 < 4$ then statement is true $\Rightarrow$ A also true (*)

D $\quad$ The statement (*) is true if and only if $x^2 < 4$ $\Rightarrow$ D + C true $\quad \times$

Given that exactly one of these options is correct, which one is it?



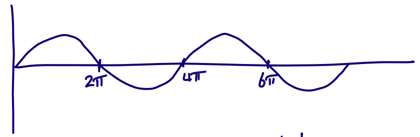

C: If inside ◯ then statement is true
$\Rightarrow$ A is true as ◯ is enclosed by ◯

A: If inside ◯ then statement is true
$\Rightarrow$ C may not be true if in ◯

14. Which of the following is a necessary and sufficient condition for

$$\int_0^{k\pi} sin\left(\frac{x}{2}\right)dx = 0$$

$y = \sin\left(\frac{x}{2}\right)$




If $\int_0^{k\pi} y\,dx = 0$ then $k$ must be a multiple of 4

If $k$ is multiple of 4 then $\int_0^{k\pi} y\,dx = 0$

A $\quad k = 2$ ×

B $\quad k = 4$ sufficient, not necessary (eg $k = 8$)

C $\quad k$ is a positive integer ×

D $\quad k$ is a multiple of 2 ×

E $\quad k$ is a multiple of 4

15. Which of the following is a sufficient but not necessary condition for

$$\sum_{k=1}^{n} cos\left(\frac{k\pi}{4}\right) = 0$$

If condition then sum ⇒

$\cos\frac{\pi}{4} + \cos\frac{\pi}{2} + \cos\frac{3\pi}{4} + \ldots$

$= \underbrace{\frac{\sqrt{2}}{2} + 0 + \left(-\frac{\sqrt{2}}{2}\right)}_{\Sigma = 0} + (-1) + \left(-\frac{\sqrt{2}}{2}\right) + 0 + \frac{\sqrt{2}}{2} + 1 + \ldots$

$\Sigma = 0$

sum $= 0$ for $n = 3, 8, 11, 16, \ldots$

A $\quad n = 2$ ×

B $\quad n = 4$ ×

C $\quad n$ is 3 more than a multiple of 4 ×

D $\quad n$ is 3 more than a multiple of 8 sufficient, not necessary

E $\quad n$ is an even integer ×

F $\quad n$ is an odd integer ×

16. Which of the following is a necessary but not sufficient condition for the curve

$y = x^2(x - k)$ to have a local minimum point at $x = \frac{2}{3}k$

If ↗ then ↘

A $\quad k > 0$ N+S

B $\quad k \geq 0$ N

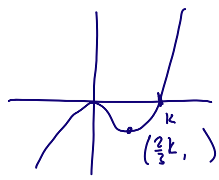
C $\quad k < 0$ ×

D $\quad k \leq 0$ ×

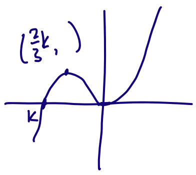
E $\quad k = 1$ S

F $\quad k = -1$ ×

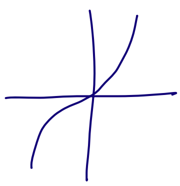
$k > 0$




$k < 0$




$k = 0$



$y = x^3 - kx^2$

$y' = 3x^2 - 2kx = 0$

$x = 0 \quad x = \frac{2}{3}k$

$y'' = 6x - 2k$

At $x = \frac{2}{3}k$

$y'' = 2k \quad \therefore$ max when $k < 0$, min when $k > 0$

$\therefore k > 0$ necessary + sufficient

$k \geq 0$ necessary but not sufficient (when $k = 0$)

**Quantifiers**

The words 'all', 'some, 'none' are examples of quantifiers. These tell us how many instances satisfy the statement, and a statement containing one or more of these words is called a **quantified statement**.

In English there are many ways to write these statements:

All: All / Every / Each / Any / If

**All** even numbers are divisible by 2

**For all** even numbers $x$, $x$ is divisible by 2

**For each/every** even number $x$, $x$ is divisible by 2

**If** $x$ is an even number, **then** $x$ is divisible by 2

Some: Some / At least one / There exists

**Some** even numbers are prime

**For some** even number $x$, $x$ is prime

**There exists** an even number $x$, such that $x$ is prime

**There is at least one** even number $x$, for which $x$ is prime

None: No / Not any / There does not exist / There are no

**No** real square numbers are negative

There are **not any** real square numbers that are negative

There **does not exist** a real square number $x$ for which $x$ is negative

There **is no** real square number $x$ such that $x$ is negative

Consider the order of quantifiers:

When the same type of quantifier is used, the order does not matter:

**For all** odd numbers x and **all** even numbers $y$ the sum of $x$ and $y$ is odd

**For all** even numbers $y$ and **all** odd numbers $x$ the sum of $x$ and $y$ is odd

However, with different quantifiers, the order changes the meaning of the statement:

**For all** positive real $x$, **there exists** a real $y$ such that $y^2 = x$

This is TRUE as we can choose any positive value for x and find a value of y that makes the equation true by calculating $y = \sqrt{x}$ .

**There exists** a real $y$, such that **for all** positive real $x$, $y^2 = x$

However this is FALSE as there is not a single value of y that makes the equation true; the value of $y$ that we need changes with our choice of $x$.

1. Are the following **quantified statements** true or false?

a) For all odd integers $x$, the quantity $x^2 + 6x + 9$ is even True

odd + even + odd = even

b) There exists an even integer $y$, such that the quantity $y - 8$ is prime True

When $y = 10$ $y - 8 = 2$ is prime

c) For all prime integers $x$, the quantity $x^2 + x + 11$ is prime False

When $x = 11$ $11^2 + 11 + 11 = 11(11 + 1 + 1)$ not prime

d) For all positive real values of $x$, we have $x^2 \geq x$ False

When $x = \frac{1}{2}$ $x^2 = \frac{1}{4} \leq \frac{1}{2}$

e) For any integers $a$, $b$, and $c$, if $a$ divides $bc$, then $a$ divides $b$ or $a$ divides $c$ False

When $a = 6$ $b = 9$ $c = 8$

f) For all real $x$, $\dfrac{x}{x} = 1$ False when $x = 0$

(statement true for $x \neq 0$)

g) For all $x > 0$, there exists $y$ such that $x + y = 0$ True

Pick $y = -x$

h) There exists a real value of $y$ such that $x + y = 0$ for all $x > 0$ False

Need to choose different $y$ for each choice of $x$

i) There is no real $x$, such that $x^2 = 3$ False

choose $x = \pm\sqrt{3}$

j) There exists a real $x$ and a real $y$ such that $x \leq y$ True

choose any pair eg $x = 1$ $y = 2$

k) For all real $x$ there exists a real $y$ such that $xy = 1$ False

when $x = 0$ there is no value of $y$

This statement is true for $x \neq 0$ : choose $y = \frac{1}{x}$

**Negation (denial not opposite)**

The negation of a statement is achieved by placing ‘not’ in front of the statement. In reality there are often multiple ways of phrasing this in English. Be careful not to infer too much from a negation, for example ‘not hot’ does not mean cold - it just means not hot (eg it could be warm).

If our original statement is True, then the negation will be False and vice versa.

| **Statement** | **Negation** |
|---|---|
| He is a doctor | Not ‘He is a doctor’ = He is not a doctor |
| She is tall | Not ‘She is tall’ = She is not tall (She is short would be incorrect) |

1. I am hungry — I am not hungry
2. They do their homework — They don't do their homework
3. It is not raining — It is raining
4. The melon is not ripe — The melon is ripe

---

To negate *(A and B)* we use *not (A and B)* which is the same as *not A or not B*

I have blue eyes **and** blond hair — Either I do not have blue eyes **or** I do not have blond hair (**or** I do not have either)

5. My socks are blue and stripy — Either not blue or not stripy
6. I play hockey and basketball — Don't play both – either not hockey or not basketball
7. I had lunch with Bill and Ben — Didn't have lunch with both / Both didn't have lunch with me
8. Neither my brother nor sister will help me — Either my brother or my sister will help me

---

To negate *(A or B)* we use *not (A or B)* which is the same as *not A and not B / neither A nor B*

I study English **or** German — I do not study English **and** I do not study German

9. Jan drinks tea or coffee — Jan doesn't drink tea and doesn't drink coffee
10. The man is called Jim or John — not called Jim and not called John
11. The children eat apples or bananas — The children eat neither apples nor bananas
12. It is not hot and not sunny — It is hot or sunny

---

*Note how the words ‘**and**’ & ‘**or**’ swap when we negate a statement*

To negate *(for all A, then B)* we use *not (for all A, then B)* which is the same as *not every A implies B / there exists A such that not B*

| **Statement** | **Negation** |
|---|---|
| Everyone like pizza | **Not** everyone likes pizza / At least one person doesn't like pizza / Some people don't like pizza / There exists someone who doesn't like pizza |

13. All vegetarians eat carrots — Some vegetarians don't eat carrots

14. My teacher is always right — My teacher is sometimes wrong

15. All dogs bark — Some dogs don't bark

16. Not every integer is odd — All integers are odd

---

To negate *(there exists A such that B)* we use *not (there exists A such that B)* which is the same as *there is no A such that B / for all A, not B*

| | |
|---|---|
| There is a prime number less than 2 | There are no prime numbers less than 2 <br> All prime numbers are greater than or equal to 2 |

17. Some boys like football — No boys like football / All boys don't like football

18. At least one square number is less than 3 — No square numbers are less than 3

19. There exist some birds who can not fly — All birds can fly

20. There are no prime numbers that are even (FALSE Statement) — At least one prime number is even (True Statement)

---

To negate *(if A, then B)* we use *not (if A, then B)* which is the same as *if A, then not B / A and not B*

| | |
|---|---|
| If the sun shines, I will wear a hat | If the sun shines, I will not wear a hat |

21. If it is raining I will take an umbrella — If it is raining, I won't take an umbrella

22. I will receive a gold medal if I win — If I win, I won't receive a gold medal

23. If $a < b$ then $f(a) < f(b)$ — If $a < b$ then $f(a) \geq f(b)$

24. $f(a) > 0$ if $a > 0$ — if $a > 0$ then $f(a) \leq 0$
(= if $a > 0$ then $f(a) > 0$)

---

*Note how the words '**for all**' & '**there exists**' swap when we negate a statement*

**Negation of 'Nested' statements**

**A** Consider the statement: The class can complete their homework online, if for **every** student in the class, the student has online access

The negation is: A class can **not** complete their homework online, if there is **at least** one student who does **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has online access

Then the statement becomes: P is true if **for every** Q, **there exists** R

The negation of this is: P is **not** true if **there exists** Q such that **not** R

**B** Consider the statement: The class can complete their homework online, if **for every** student in the class, the student **has** a friend who has online access

The negation is: A class can **not** complete their homework online if **there exists** a student in the class, **all** of whose friends do **not** have online access.

Replace parts of this statements as follows: P = class can complete homework online

Q = student in the class

R = student has a friend

S = friend has online access

Then the statement becomes: P is true if **for every** Q, **there exists** 'R such that S'

The negation of this is: P is **not** true if **there exists** Q such that **not** '**there exists** R such that S'

or: P is **not** true if **there exists** Q such that '**for all** R **not** S'

---

Write the negation of the following nested statements:

1) A set of integers $P$ is the set of even numbers if and only if for any integer $n$ in $P$, $\frac{n}{2}$ is also an integer.

P iff for any B then C

not P if there exists B such that not C

P not set of even numbers if there exists an integer n in P such that $\frac{n}{2}$ is not an integer

2) A set of integers $P$ is the set of square numbers if and only if for any integer $n$ in $P$, there exists an integer $k$ such that $k^2 = n$

P iff for any B there exists C such that D

not P if there exists B such that not 'exists C such that D'

———" ——— all C not D

P not set of square numbers if there exists an integer n in P such that for all integers k, $k^2 \neq n$

**Counter Examples**

A counter example is one example which disproves a statement. It proves a statement is not true.

1) Find a counter example to the following statements:

a) All quadrilaterals with equal side length are squares Rhombus

b) The square root of a number is always less than the number $\sqrt{\frac{1}{4}} = \frac{1}{2}$

c) If a three-dimensional solid has a circular base, then it is a cylinder Cone

d) If $n$ is an integer and $n^2$ is divisible by 4, then $n$ is divisible by 4 $n = 6$

e) If $p$ is an odd prime then $p+2$ is also an odd prime $p = 7$

f) The sum of 2 numbers is always greater than both numbers $(-2) + (-6)$

g) $10k^2 + 1$ is prime if $k$ is an odd prime $k = 3$ $10(3)^2 + 1 = 91 = 7 \times 13$

h) For all real $x$, $5x > 4x$ any $x < 0$

i) For all real $x$, $\sqrt{1 - sin^2x} = cosx$ $x = 180$ $(1 = -1)$

2) A set of five signs has a letter printed on the left and a number printed on the right

A 8 B 4 C 1 D 7 E 3

Which sign(s) provide a counterexample to the following statements:

a) Every card that has a vowel on the left has an even number on the right E3

b) Every card that has an even number on the right has a vowel on the left. B4

c) Every card that has a consonant on the left has a prime number on the right B4 C1

d) Every card that has a prime number on the right has a vowel on the left D7

3) How many counter examples are there to the following statements:

a) All odd numbers between 2 and 20 are prime. 3 5 7 9 11 13 15 17 19

2 counterexamples (odd but not prime)

b) If n is a prime integer less than or equal to 10, then $n^2 + 2$ is also prime

$n =$ 2 3 5 7

$n^2+2 =$ 6 11 27 51

3 counterexamples ($n$ prime $< 10$, $n^2+2$ not prime)

c) A whole number n less than or equal to 50, is prime if it is 1 less or 5 less than a multiple of 6

1 less or 5 less : 1 5 7 11 13 17 19 23 25 29 31 35 37 41 43 47 49

4 counterexample (1 less or 5less, not prime)

4) Prove the following statements, or find a counter example to disprove them:

a) For all real $x$, $3x < 3^x$

False : when $x = 1$ $\quad 3 < 3$ is not true

b) For all real $x$, if $ax = bx$ then $a = b$

False : when $x = 0$ $\quad a, b$ can be different

c) If a positive integer $p$ has remainder 1 when divided by 3, then $p^3$ also has remainder 1 when divided by 3

True : $p = 1, 4, 7 \ldots \quad p^3 = 1, 64, 343, \ldots$

$p = 3n + 1 \quad p^3 = 27n^3 + 9n^2 + 3n + 1 = 3(9n^3 + 3n^2 + n) + 1$

d) For all integers $p$ and $q$, if $p < q$, then $p^2 < q^2$

False : $p = -5, \; q = 1$

e) For all integers $n$, $9n^2 + 24n$ is not prime

True : $9n^2 + 24n = 3(3n^2 + 8n)$ has factor 3 for all $n$

f) If a positive integer $p$ is prime, then $2p + 1$ is also prime

False : $p = \; 2 \; 3 \; 5 \; 7$

$2p + 1 = \; 5 \; 7 \; 11 \; 15$ $\qquad p = 7$

g) For consecutive even integers $p$ and $q$, $p^3 - q^3$ is a multiple of 8

True: $p = 2k + 2 \quad q = 2k$

$p^3 - q^3 = 2^3(k+1)^3 - 2^3k^3$ which is a multiple of $2^3 = 8$

h) For all integers $n$, if $n$ is prime, then $(-1)^n = -1$

False : $n = 2 \quad (-1)^2 = 1$

i) $4^n + 3^{n-2} + 3$ is divisible by 5 for all integers $n \geq 2$

False : $n = 2 \quad 16 + 1 + 3$ ✓

$n = 3 \quad 64 + 3 + 3$ ✓

$\underline{n = 4} \quad 256 + 9 + 3$ ✗ (last digit not 0 or 5)

j) If $p$ and $q$ are irrational, such that $p \neq q$, then $p + q$ is irrational

False : $p = +\sqrt{2} \quad q = -\sqrt{2} \quad p + q = 0$ rational

k) If $p$ is rational and $q$ is irrational, then $log_p q$ is irrational

False : $p = 2 \quad q = \sqrt{2} \quad \log_2 \sqrt{2} = \frac{1}{2} \log_2 2 = \frac{1}{2}$ rational

**Logic**

On an island people either always tell the truth or always tell lies. Identify the truth-tellers or liars in the following situations.

1. You are approached by 2 people, **A** and **B**.

a) **A** says "we both always tell lies"

If A truth ⇒ both tell lies × contradiction

∴ A liar ⇒ not both tell lies ⇒ B truth ✓

b) **A** points to **B** and says "he is a liar" and **B** says "neither of us are liars"

If A truth ⇒ B is liar ⇒ at least one liar ✓

If A liar ⇒ B truth ⇒ no liars × contradiction

c) **A** says "we are both telling the truth" and **B** replies "he is lying".

If A truth ⇒ B truth ⇒ A liar × contradiction

A liar ⇒ at least one liar

If B liar ⇒ A truth × contradiction If B truth ⇒ A liar ✓

d) **A** says "at least one of us is lying"

If A truth ⇒ B liar ✓

If A liar ⇒ no liars × contradiction

e) **A** says "exactly one of us is lying", and **B** replies "actually we're both lying"

If A truth ⇒ B liar ⇒ at least one truth ✓

If A liar ⇒ either no liars or both liars ⇒ both liars

but B liar ⇒ at least one truth

× contradiction

2. You are approached by 3 people, **C**, **D** and **E**.

**C** says "**D** is a liar". **D** says "**E** is a liar". **E** says "**C** and **D** are both liars".

If C truth ⇒ D liar ⇒ E truth × contradiction

If C liar ⇒ D truth ⇒ E liar ✓

3. You are approached by 2 people, **F** and **G**.

**F** says "Either I am a liar or **G** is a truth-teller". (or both)

If F truth ⇒ G truth ✓

If F liar ⇒ F is not a liar and G is not a truth teller

× contradiction

4. You are approached by a group of 4 people, **H**, **J**, **K**, and **L**.

**J** says "**K** is a liar". **K** says "**L** is a truth-teller".

**L** says "**J** never tells the truth". **H** says "**K** is a liar and I am a liar".

If J truth ⇒ K liar ⇒ L liar ⇒ J truth H liar × contradiction

H truth × contradiction

If J liar ⇒ K truth ⇒ L truth ⇒ J liar H truth × contradiction

H liar

OR H truth – contradiction

H liar ⇒ K truth ⇒ L truth ⇒ J liar ✓

**Errors in Proofs**

Common errors to look out for include dividing by zero; only taking the positive square root; arithmetic or algebraic errors; incorrect manipulation of minus signs; minus signs in inequalities; decreasing functions in inequalities; finding all trig solutions; invalid conclusions.

1. Find the error(s) in the following attempted solutions:

a)

$$2log_3 x - log_3\sqrt{x} = 5$$

incorrect use of log law

$$2log_3\frac{x^2}{\sqrt{x}} = 5$$

$$2log_3\sqrt{x} = 5$$

$$log_3 x = 5$$

incorrect conversion to index form

$$x = 5^3 = 3^5$$

b)

$$\frac{4x+1}{x-1} < 3$$

(x-1) could be negative so can't multiply by (x-1)

$$4x + 1 < 3(x - 1)$$

incorrect expansion

$$4x + 1 < 3x - 1$$

$$x < -2$$

c)

$$4sin^2x = 3tan^2x$$

need ± when taking the square root

$$2sinx = \sqrt{3}tanx$$

$$2sinx = \sqrt{3}\frac{sinx}{cosx}$$

$$2cosx = \sqrt{3}$$

dividing by sinx loses the solution sinx = 0

$$cosx = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3}$$

wrong value $\left(\frac{\pi}{6}\right)$ and only principal value given

d)

$$\frac{x}{x+1} - \frac{x-1}{x+2} = 0$$

$$\frac{x(x+2) - (x+1)(x-1)}{(x+1)(x+2)} = 0$$

$$x(x+2) - (x+1)(x-1) = 0$$

incorrect expansion

$$x^2 + 2x - x^2 - 1 = 0$$

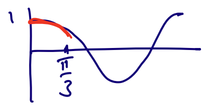
$$2x = 1$$

$$x = \frac{1}{2}$$

e) Given $0 \le x \le \frac{\pi}{3}$ find the range of values for $cosx$

$0 \le x \le \frac{\pi}{3}$

$\frac{1}{2} \le cos x \le 1$




$$x \ge 0 \qquad\qquad x \le \frac{\pi}{3}$$

$$cosx \ge 0 \qquad\qquad cosx \le cos\frac{\pi}{3}$$

No $-1 \le cos x \le 1$

$$cosx \le \frac{1}{2} \qquad\qquad 0 \le cosx \le \frac{1}{2}$$

No as cosine is a decreasing function for $0 \le x \le \frac{\pi}{3}$

Find the error(s) in the following proofs and circle the line numbers where these occur:

2) Let $x = 1$

| | | |
|---|---|---|
| multiply by $x$ | $x^2 = x$ | (I) |
| subtract 1 | $x^2 - 1 = x - 1$ | (II) |
| factorise | $(x+1)(x-1) = x - 1$ | (III) |
| divide by $(x - 1)$ | $x + 1 = 1$ | (IV) dividing by $x-1=0$ |
| substitute $x = 1$ | $2 = 1$ | (V) |

3) Let $a, b$ be two numbers where $a \neq b$ and let $a + b = 2c$

| | | |
|---|---|---|
| multiply by $(a - b)$ | $a^2 - b^2 = 2ac - 2bc$ | (I) |
| add $b^2 + c^2 - 2ac$ | $a^2 - 2ac + c^2 = b^2 - 2bc + c^2$ | (II) |
| factorise | $(a - c)^2 = (b - c)^2$ | (III) |
| take the square root | $a - c = b - c$ $\quad a-c = -b+c$ | (IV) take $\pm\sqrt{\ }$ |
| so | $a = b$ $\quad a+b = 2c$ | (V) |

4) Consider the attempt at solving the following equation

$$sin^2 A + 1 = 2 - cos^2 B$$

| | | |
|---|---|---|
| rearrange | $sin^2 A = 1 - cos^2 B$ | (I) |
| substitute for $cos^2 B$ | $sin^2 A = sin^2 B$ | (II) |
| take the square root | $sin A = sin B$ | (III) take $\pm\sqrt{\ }$ |
| inverse sine both sides | $A = B$ | (IV) |

or $A = \pi - B$ etc
$A = 2\pi + B$

5) Given that $N = k^2 - 1$ and $k = 2^p - 1$ where $p$ is an integer, show that $2^{p+1}$ is a factor of $N$

| | | |
|---|---|---|
| Consider | $2^{p+1}(2^{p-1} - 1)$ | (I) |
| expand | $= 2^{2p} - 2^{p+1}$ | (II) |
| use law of indices | $= 2^{2p} - 2(2^p)$ | (III) |
| complete the square | $= (2^p - 1)^2 - 1$ | (IV) |
| substitute for $k$ | $= k^2 - 1$ | (V) |
| substitute for $N$ | $= N$ | (VI) |

Therefore $2^{p+1}$ is a factor of $N$

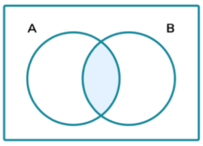
This is a proof of the converse "If $2^{p+1}$ is a factor of $N$..."

# TMUA Proof and Logic Summary using Venn Diagrams

Definitions

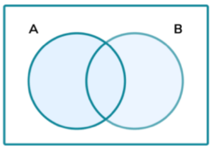
**and** $A$ **and** $B$ means $A$ and $B$ together ($A \cap B$)

For $A$ **and** $B$ to be true, **both $A$ and $B$** must be true




**or** $A$ **or** $B$ means $A$ or $B$ or both ($A \cup B$)

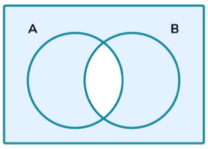
For $A$ **or** $B$ to be true, **either $A$ or $B$ or both** must be true




negation = **not**

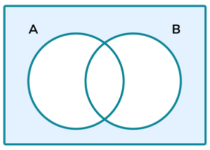
**not ($A$ and $B$) = not $A$ or not $B$**

***not*** *(blue eyed* ***and*** *blonde) = not blue-eyed or not blond*
so could be one or the other but not both




**not** ($A$ **or** $B$) = **not** $A$ **and not** $B$

*not (blue eyed* ***or*** *blonde) = not blue-eyed and not blond*
so does not have either characteristic

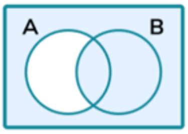



**if, then**

if $A$ then $B$ means if $A$ is true, then $B$ must be true (But if $A$ is not true, then $B$ could be true or false)

We can also write this in the following ways:

| | |
|---|---|
| A implies B | $A \implies B$ |
| B if A | A only if B |
| B is **necessary** for A | A is **sufficient** for B |




The **converse** statement (swapping statements) is

'if B then A' but these are not always equivalent

The **contrapositive** statement (swapping and negating both statements) is

'if not B then not A' and this is an equivalent statement to the original.

The **negation** of the statement is

'if A then not B' or 'A and not B'

**if and only if**

$A$ if and only if $B$. We can also write this in the following ways:

| | |
|---|---|
| $A$ implies $B$ and $B$ implies $A$ | $A \iff B$ |
| $A$ if B and A only if B | $A$ iff $B$ |
| $A$ is **sufficient** and **necessary** for $B$ | |