corrected 10·8·26

Combinatorics

Combinations and permutations; choose function; distinct / indistinguishable ; distribution into bins; distributing part of a set; stars and bars method; repetition and restrictions.

Combinations

A collection of things where the order is not relevant, so ABC is the same as BCA. You can think of this as a group - examples include the toppings on a pizza, or choosing a sports team.

Assume we have n distinct objects, and we want to choose r objects,

then the number of combinations is n *choose* r which can be written as:

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{where } n! = n(n-1)\ldots 3\times 2\times 1$$

1. Five children are travelling in a car with three seats in the middle row and two seats in the back row. How many ways are there to choose which two of the five children sit in the back row?

$$\binom{5}{2} = \frac{5\times4\times3\times2\times1}{(2\times1)\times(3\times2\times1)} = \frac{5\times4}{2} = 10$$

2. A restaurant has a buffet consisting of eight distinct dishes. How many ways are there to choose a different pair of dishes?

$$\binom{8}{2} = \frac{8\times7}{2} = 28$$

3. Greg has a collection of 12 toy trains. He wants to display five of them in a display case. How many different ways can he choose five trains to display?

$$\binom{12}{5} = \frac{\cancel{12}\times11\times\cancel{10}\times9\times8}{5\times4\times3\times2} = 792$$

4. There are 18 athletes in a squad including the captains Anna and Ben. How many ways can a team of four be chosen, that includes both captains?

choose captains, then 2 from 16 $\quad \binom{16}{2} = \frac{16\times15}{2} = 120$

Permutations

A collection of things where the order is relevant, so **ABC** is different to **BCA**. You can think of this as a list - examples include the digits on a padlock, or prize winners in a competition.

Assume we have n distinct objects, and we want to choose r objects:

Case 1: repeats allowed (eg choosing a 3-digit number where 224 or 666 is allowed)

then the number of arrangements is written as: n^r

Case 2: repeats not allowed (eg only one of each object - once it is chosen it can't be chosen again)

then the number of arrangements is n *permute* r and is written as:

$$^nP_r = \frac{n!}{(n-r)!} \qquad \text{where } n! = n(n-1)\ldots 3\times 2\times 1 \quad \text{and} \quad 0! = 1$$

Case 3: some repeats, so we have n objects, where p are the same, and q are the same, and r are the same, (etc), then the number of arrangements is: $\dfrac{n!}{p!\times q!\times r!}$

1. How many 4 digit codes can be made using any digits from 1 to 9.

$9\times 9\times 9\times 9$ or 9^4

2. How many 4 digit codes can be made using different digits from 1 to 9.

$9\times 8\times 7\times 6$ or $\dfrac{9!}{5!}$

3. Filip wants to make a sign from five cards with the letters, A A N T T
How many ways can he arrange the cards in a distinct order?

$\dfrac{5!}{2!\,2!} = \dfrac{120}{2\times 2} = 30$

repeating A's

repeating T's

4. The digits 1, 3, 5 and five copies of the digit 9 are arranged to form an 8-digit integer.
How many different integers can be formed?

$\dfrac{8!}{5!} = 336$

5. How many ways are there to arrange the letters of the word BOTTLES such that both of the vowels are at the ends?

$2\times \dfrac{5!}{2!} = 120$

2 ways to have vowels at each end

Distributions
A distribution of objects into sets, is an arrangement of those objects, such that each object is placed into one of the sets.

Distributing **Distinct Objects** into **Distinct Sets**
In this case each object is different - for example 5 numbered balls - or 3 types of fruit.
Each set is different - for example - 4 flavours of crisps - or 6 coloured bags.

Say we have n objects to be distributed into r sets.
There are r options for each object, so the number of distributions is r^n.

Example - 5 coloured balls are to distributed amongst 3 coloured bags. The first ball can be placed in 3 possible ways. Similarly for the second ball, etc. So the total number of ways is $3 \times 3 \times 3 \times 3 \times 3 = 3^5$

Distributing Part of a Set of Objects

Say we have n objects and we want to distribute k of these objects into r sets.

There are ${}^nC_k$ ways of choosing these k objects, and for each of these choices, we have k objects to be distributed into r sets (as above), so the number of distributions is ${}^nC_k\, r^k$

Example - a teacher has a box of 8 different coloured pencils. She is going to choose 6 of these pencils to give to 5 children in her class. How many different ways can she do this?

${}^8C_6 = 28$ ways to choose 6 pencils from 8

5^6 ways to allocate the 6 pencils to 5 children Total = 28×5^6

1. Three friends Alf, Beth and Clare are each going to choose one flavour of ice-cream from the following flavours: vanilla, chocolate, strawberry, coffee and mint.
How many ways are there for the friends to choose their ice-creams?

5^3

Alf has 5 options
Beth has 5 options
Clare has 5 options

2. There are eight different novels and three sisters who want to read them. How many ways are there to distribute the novels between the sisters.

3^8

First novel - 3 options
Second novel - 3 options ...

3. There are seven different flavoured cakes for sale in a bakery, and six people waiting in the queue. The first person buys five of the cakes for a party.
What is the total number of ways that the seven cakes can be sold?

Choose 5 of the 7 cakes
$\binom{7}{5} = 21$ ways

2 left → 5 people $= 5^2$ ways
$\therefore 21 \times 25 = 525$

4. 10 people buy tickets for a raffle with six different prizes. The first winner can choose three of the prizes. The second winner can choose two of the remaining prizes, and the third winner gets the final prize. How many ways are there for the prizes to be distributed?

1st - 3 prizes   2nd - 2 prizes
$10 \times \binom{6}{3} \times 9 \times \binom{3}{2} \times 8$
$10 \times 20 \times 9 \times 3 \times 8 = 43200$

Identical Objects into Distinct Sets (Stars and Bars Method)

Example: A grandmother has seven cookies to distribute to her four grandchildren.
a) How many ways can she distribute the cookies?

Represent the seven identical cookies with 'stars' * * * * * * *

Represent the four distinct children A, B, C, D as four sets created by three 'bars' A | B | C | D

So the following pattern * | * * * | * * * |

means that child A gets 1 cookie, child B gets 3 cookies, child C gets 3 cookies and child D gets none.

So the problem becomes how many ways can we arrange 7 stars and 3 bars, or how many ways can we choose 3 positions for the bars out of 10 possible positions

Answer $^{10}C_3 = 120$

b) How many ways can she distribute the cookies so that every child gets at least one cookie?

With this extra condition, we first allocate one cookie to each child. We then have three cookies left and repeat the method above with three stars (for the cookies) and three bars (for the four children)

Answer $^{6}C_3 = 20$

This powerful method can be used in many different ways, which at first glance may not be obvious:

Example 1:
How many ordered sets of non-negative integers are there such that $a + b + c + d = 10$

If we assign the value '1' to each star, then 10 stars can be used to represent this sum.

We then need to divide the stars into 4 groups to represent the 4 digits, using 3 bars.

So the following pattern * | * * * | * * * | * * * represents 1+3+3+3

The problem becomes how to choose 3 positions from 13 $= {}^{13}C_3$

Example 2:
A positive integer has 5 digits, and these digits sum to 4. How many possible integers are there?

If we assign the value '1' to each star, then 4 stars can be used to represent this sum.

We then need to divide the stars into 5 groups to represent each digit, using 4 bars.

If we accept that the first digit can not be zero, then we know we have to start the sequence with a star. So we are left with arranging 3 stars and 4 bars.

So the following pattern * | | * | * * | represents 10120

The problem becomes how to choose 4 positions from 7 $= {}^{7}C_4$

Black | Blue | Red   need 2 bars

1. Three pirates Blackbeard, Bluebeard, and Redbeard are sharing treasure of 8 identical gold coins. How many ways can they share the treasure between themselves?

8 * 2 | $\binom{10}{2} = 45$

2. There are six distinct fish in a tank. Ten pellets of food are floating on the surface. How many ways are there for the fish to eat the food?

10 * 5 | $= \binom{15}{5} = \frac{\cancel{15}^{3} \times \cancel{14}^{7} \times 13 \times \cancel{12} \times 11}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = 91 \times 33$

max 3 digits   eg.

3. A positive integer has less than 4 digits, and these digits sum to 4. * || * * * = 103
How many possible integers are there?

4 * 2 | $\binom{6}{2} = 15$

+ve or zero

4. Find the number of non-negative integer solutions to $a + b + c + d + e = 16$

16 * 4 | $\binom{20}{4}$

5. You have 6 identical balls to be placed into 6 numbered boxes.
How many ways can the balls be distributed if some boxes can be empty?

6 * (balls) 5 | (representing 6 boxes) $\binom{11}{5} = \frac{11 \times \cancel{10} \times \cancel{9}^{3} \times \cancel{8}^{2} \times 7}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}} = 462$

6. An eight digit number has digits that sum to 4. How many possible numbers are there?

4 * but 1st digit ≠ 0 ⇒ Put 1* at far left. Then arrange 3 * 7 |
$\binom{10}{7} = \frac{10 \times 9 \times 8}{6} = 120$

one * in each group to start

7. Find the number of positive integer solutions of the equation $x + y + z = 12$

No 0's 9 * 2 | $\binom{11}{2} = 55$

8. Four identical badges are to be distributed to three children such that each child receives at least one badge. How many ways can they be distributed?

1 each ⇒ 1* 2| $\binom{3}{2} = 3$

9. There are twelve slices of pizza being shared by a group of 5 friends?
How many ways can they share the pizza such that everyone receives:
a) at least one slice?
b) at least two slices?

a) 7 * 4 | $\binom{11}{4}$
b) 2 * 4 | $\binom{6}{4}$

10. Jane buys 28 pencils for her class. The shop has red, blue or green pencils for sale.
How many ways can she select the 28 pencils?

28 * 2| $\binom{30}{2}$

11. Max is making gift bags with toys in them. The party shop sells 4 different toys and Max is making 9 bags, each containing a single toy. How many different combinations can Max make?

9 * 3 | $\binom{12}{3}$

12. There are 6 boys and 10 girls in a line.
How many ways are there to arrange them such that no 2 boys are next to each other?

Fix Boys $B_G B_G B_G B_G B_G B$ Put 1G between each
5 remaining girls 7 positions $\binom{11}{5} = 462$
5* = 6 |

13. 3 red balls and 12 blue balls are randomly placed in a row.
What is the probability that no two red balls are adjacent?

Fix R B R B R leaves 10 blue in 4 places
$\binom{15}{3} = 455$ 10 * 3 | $\binom{13}{3} = 286$

Problems with Restrictions

1. A 4-digit number is created from 4 distinct integers chosen from {1,2,3,4,5,7,9}
   How many numbers include both 1 and 2 but neither begin nor end with 1? 1, 2, a, b

   $\binom{5}{2} = 10$
   choose final 2 digits from {3, 4, 5, 7, 9}

   $2 \times 3 \times \underbrace{2 \times 1}_{\text{last 2 digits}} = 12$
   '1' can go in 2 places
   '2' can go in 3 remaining places

   $10 \times 12 = 120$

2. From a group of 11 scouts, including Victor and Oscar, 4 people are selected to stand in the front row for a photograph. If Victor and Oscar are both in the front row, how many ways can the front row be arranged?

   $\binom{9}{2} \times 4! = 864$

3. How many ways can the integers 1, 2, 3, 4, 5, 6 be arranged such that 2 is adjacent to either 1 or 3 or both?

   $\boxed{12}$ $2 \times 5! = 240$

   $\boxed{13} = 240$

   $480 - 48 = 432$

   $\boxed{123}$
   $\boxed{321}$

   $2 \times 4! = 48$
   subtract as double counting

4. Ten players enter a badminton competition. The first round consists of 5 matches, with each player in one match. How many ways could the 10 players be matched against each other?

   $\binom{10}{2} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2} = 45 \times 28 \times 15 \times 6$
   first match

5. How many ways are there to pick two distinct numbers from 1 to 11 such that the sum of the two numbers is even?

   Both odd $\binom{6}{2} = 15$
   or
   Both even $\binom{5}{2} = 10$
   + } 25

6. There is a group of people in a room. They each shake hands with every other person (but not themselves!). If there are 66 handshakes, then how many people are in the room?

   n people $\binom{n}{2} = \frac{n(n-1)}{2} = 66$ $n(n-1) = 132$

   $n = 12$

Mixed Questions

1. Ed, Fred Jed and Ned are going to stand in a line to have their photo taken. How many ways are there for them to arrange themselves?

$4 \times 3 \times 2 \times 1 = 24$

2. During a game of hide and seek, four children are hiding in a house with seven possible rooms for them to hide in. Each child hides in a different room. How many ways are there for the children to hide?

$7 \times 6 \times 5 \times 4 = 840$ OR $\binom{7}{4} \times 4! = 840$

3. How many ways are there for six friends to arrange themselves around a round table (ignoring rotations of the same arrangement?)

fix first person to account for rotations : $5! = 120$

4. There are 20 football players in a squad including the joint captains Jo and Kim. How many ways can a team of eleven be chosen, that includes both captains?

choose captains, then 9 from 18 $\binom{18}{9} = \frac{18!}{9!\,9!}$

5. Consider all 5 letter 'words' made from the letters *a* to *h*

a) How many words are there in total? 8^5

b) How many words contain no repeated letters? $8 \times 7 \times 6 \times 5 \times 4$

c) How many words start with the sub-word '*aha*' 8^2 (other 2 letters)

d) How many words contain the subword '*bad*' 3×8^2

e) How many words with no repeated letters contain the subword '*bad*'

$3 \times 5 \times 4 = 60$

6. How many two letter words start with one of the 5 vowels?

$5 \times 26 = 130$

7. You have 3 different flavours of cake, 4 colours of icing, and 2 types of sprinkles. How many different types of cupcake can you make?

$3 \times 4 \times 2 = 24$

8. How many licence plates can you make out of three letters followed by three numerical digits?

$26^3 \times 10^3$

9. Consider the five letters of the word 'M A T H S' written on five separate cards.

These cards are arranged to make five letter 'words'.

a) How many 'words' can be made using these cards? $5! = 120$

b) How many 'words' start with M $4! = 24$

c) How many 'words' have the letters S and H next to each other $4! \times 2 = 48$

d) How many 'words' start with M and have the letters S and H next to each other $3! \times 2$
$= 12$

10. A swimming club consists of 6 men and 4 women. A team of 6 people is selected at random to compete in a competition.

a) How many ways can a team of 6 be selected? $\binom{10}{6} = 210$

b) How many teams of 6 have an equal number of men and women?

$\binom{4}{3} \times \binom{6}{3} = 4 \times 20 = 80$

c) What proportion of teams of 6 have at least as many women as men?

4w | 2m $\binom{6}{2} = 15$ $\frac{95}{210} = \frac{19}{42}$

11. A team of five children is to be chosen at random from a group of 8 girls and 4 boys.
Find the number of possible teams containing:

a) exactly 2 girls $\binom{8}{2} \times \binom{4}{3} = 28 \times 4 = 112$

b) no more than 2 girls 1 girl: $\binom{8}{1} \times \binom{4}{4} = 8$

$8 + 112 =$

$\Rightarrow \underline{\underline{120}}$

12. An exam has two papers: the first has 8 questions of which the student must answer 5 questions.

a) Find the number of ways that the 5 questions can be chosen.

$$\binom{8}{5} = \frac{8 \times 7 \times 6}{6} = 56$$

The second paper has two sections. Section A has 9 questions of which the student must answer 6 questions, and Section B has 6 questions of which the student must answer 4 questions.

b) Find the number of ways that the student can pick questions in this paper.

$$\binom{9}{6} \times \binom{6}{4} = \frac{9 \times 8 \times 7}{6} \times \frac{6 \times 5}{2} = 1260$$

The student does not read the instructions and answers 10 questions at random from the second paper.

c) Find the probability that they chose 6 questions from Section A and 4 questions from Section B.

$\binom{15}{10} = 3003$ $\frac{1260}{3003}$

13. Three flags coloured red, yellow and blue are used to send signals. Each signal consists of one, two or three flags (where repeated flag colour is allowed).
How many distinct signals can be made?

$$3 + 3^2 + 3^3 = 39$$

14. How many ways can 3 men and 2 women be seated around a circular table, where rotations of the same arrangement are considered identical?

fix one person $\frac{4!}{3!\,2!} = 2$

15. In a group of five girls, 2 of them are wearing a red shirt. How many ways are there to seat all 5 girls in a row such that the 2 girls wearing red shirts are not sitting adjacent to each other?

$5!$ total ways $= 120$ GGG $\boxed{RR}$ $4! \times 2 = 48$ $120 - 48 = 72$

16. Three boys and 2 girls are to be seated at a round table. If the two girls want to sit next to each other, find the number of ways of seating everyone (ignoring rotations of the same arrangement).

BBB $\boxed{GG}$ Fix 1 $3! \times 2 = 12$

17. Ten people including A, B and C are waiting in a queue. How many distinct possibilities are there such that A, B and C are not all adjacent?

ABC adjacent $8! \times 3!$

$10! - 8!\,3!$

18. Five distinct numbers are chosen from the integers between 1 and 9 to make a 5 digit number. This number has 3 odd and 2 even digits. How many distinct 5-digit numbers can be created?

$\underbrace{\binom{5}{3}}_{\text{odd}} \times \underbrace{\binom{4}{2}}_{\text{even}} \times \underbrace{5!}_{\text{permutations}} = 7200$