MM7 Integration

As with the topic on differentiation, the technical integration of functions required for the TMUA is actually fairly basic, with only rational powers of x being included on the specification. (The main exception to this is integration of the modulus function, which requires first splitting the integral into two parts in order to consider negative and positive values of x separately.) However, what is required, is a good understanding of what integration means; both as the area between a curve and the x-axis, and as the reverse process of differentiation.

You should be confident with finding areas between curves and the x-axis for functions that are above or below the x-axis, and especially for functions that are a combination of both. You should understand the difference between calculating the 'definite integral' which is obtained by simply substituting in the outer limits, and finding the 'area' which might require you to consider the limits of a set of sections of the graph. Remember that area is always positive, whereas an integral can be positive or negative, or even zero!

You need to be able to determine the effect of translations, reflections and enlargements on graphs, and how these transformations affect the area(s) between the graph and the coordinate axes. Similarly you should be able to identify relationships between certain areas of graphs that are even, odd, or periodic functions, using the symmetry of these graphs.

Sometimes you might be required to compare the size of integrals of much more complicated functions, such as trigonometric or logarithmic, and here there is no actual integration required, just an ability to sketch the functions and consider the relative size of the integrals being described.

The idea that differentiation is the reverse process of integration is known as the Fundamental Theorem of Calculus and allows us to combine or separate integrals by adjusting the limits. It also leads to the following useful identity: $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

You should learn the formula for the trapezium rule and ensure you understand how it gives an over- or under-estimate for convex and concave functions respectively. You should also understand the effect of changing the number of strips being used in any interval.

Finally you may be required to solve simple differential equations with or without boundary conditions. These may be linked to other topics, such as coordinate geometry questions on the gradient function (remember the gradient is given by differentiating $f(x)$ to give $f'(x)$, so integrating the gradient function will give an expression for the original function).

Specification for TMUA

7.1 Definite integration as related to the 'area between a curve and an axis'. The difference between finding a definite integral and finding the area between a curve and an axis is expected to be understood.

7.2 Finding definite and indefinite integrals of x^n for n rational, $n \neq 1$, and related sums and differences, including expressions which require simplification prior to integrating.

7.3 An understanding of the Fundamental Theorem of Calculus and its significance to integration. Simple examples of its use may be required in the forms:

$$\int_a^b f(x)\ dx = F(b) - F(a) \quad \text{where } F'(x) = f(x) \quad \text{or} \quad \frac{d}{dx}\int_a^x f(x)\ dx = f(x)$$

7.4 Combining integrals with either equal or contiguous ranges.

7.5 Approximation of the area under a curve using the trapezium rule; determination of whether this constitutes an overestimate or an underestimate.

7.6 Solving differential equations of the form $\frac{dy}{dx} = f(x)$

MM7. Integration Worksheet - introducing key skills needed for the TMUA.

Q1 / Q2: integrate expressions containing powers of x, including negative and rational powers. Some expressions will require simplifying before integrating. Both indefinite and definite integrals.

Q3: given an equation for $\frac{dy}{dx} = f'(x)$ or the 'gradient function' use integration to find an equation for $y = f(x)$.

Q4/Q5: use integration to find the area between a curve and the x-axis, or between two curves and distinguish between areas which are above and below the x-axis.

Q6: understand how transformations affect the area under a curve and determine equivalent integrals representing these areas.

Q7: solve equations including integrals of abstract functions

Q8: use the trapezium rule to evaluate areas, and determine whether the estimate gives a value that is over or under the actual value.

Q9: use the Fundamental Theorem of Calculus to understand the relationship between integration and differentiation, especially functions $f(x)$ and their integrals $F(x)$, where

$F'(x) = f(x)$ $\qquad \frac{d}{dx}\int_a^x f(t)\ dt = f(x)$ $\qquad \int_a^b f(x)\ dx = F(b) - F(a)$

and use this link to find roots of $f(x)$, or stationary points of $F(x)$.

Q10: integrate functions such as the floor function $\lfloor x \rfloor$ ceiling function $\lceil x \rceil$, or fractional part $\{x\}$. (Definitions of these functions will be given in the question.)

1. Integrate the following expressions with respect to x

a) $\int \frac{2+5x}{3x^3} dx$

b) $\int \frac{\sqrt{x}(4-x)}{2x^2} dx$

c) $\int \frac{2x^3+\sqrt{x^3}}{\sqrt{x}} dx$

d) $\int \frac{(1+\sqrt{x})(3-\sqrt{x})}{x^4} dx$

e) $\int_{-1}^{2} |x-1|(2-x) \ dx$

2. Evaluate the following integrals

a) $\int_1^2 x^4 + 3 - \frac{2}{5x^2} \; dx$

b) $\int_1^4 \frac{x^3 + 2\sqrt{x}}{x} \; dx$

c) $\int_0^4 (x^{1/2} - 3)^2 \; dx$

d) $\int_1^5 3\sqrt{x} - \frac{1}{\sqrt{x}} \; dx$

e) $\int_0^1 \frac{15(2x+1)^2}{2\sqrt{x}} \; dx$

f) $\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}} \; dx$

g) $\int_{-2}^3 |x|(1-x) \; dx$

2. h) Given that $\int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}}\ dx = k\sqrt{3}$ where k is a constant, find k.

i) Given $\int_1^3 f(x)\ dx = \frac{4}{3}$ where $f(x) = 2x^2 + 3x + k$ and k is a constant, find k.

j) The positive number a satisfies $\int_0^a 3x^{\frac{7}{2}} + 2x^2 - 12\sqrt{x}\ dx = 0.$ Find a.

k) Find the smallest value of $I(a) = \int_0^1 (x^2 - a)^2\ dx$ as a varies

3. a) Find an equation for y, given $y = 3$ when $x = 1$

$$\frac{dy}{dx} = 6x^2 - 4x \qquad x \in \mathbb{R}$$

b) Find an equation for y, given $y = 5$ when $x = 1$

$$\frac{dy}{dx} = 4 + \frac{1}{x^2} \qquad x \neq 0$$

c) The point (-1 , -1) lies on the curve C whose gradient function is given by

$$\frac{dy}{dx} = \frac{5x^3 - 6}{x^3} \qquad x \neq 0$$ Find an equation for C.

d) The points $(0 , -3)$ and $(2 , 7)$ lie on the curve C whose gradient function is given by

$f'(x) = 3x^2 - 4x + k$ where k is a constant. Find an equation for C.

e) $f'(x) = 3 - \frac{6}{x^2}$ $x \neq 0$ Find the value of $f(2)$ given that $2f(1) - f(3) = 3$.

f) A quadratic curve C passes through (a , b) and $(2a , 2b)$, where a and b are constants.

The gradient on C is given by $\frac{dy}{dx} = 2x - 6$. Find an equation for C, in terms of a .

g) It is given that $\frac{dP}{dt} = \frac{15\pi(t-1)}{(1+\sqrt{t})}$ for t ≥ 1 and P = 3 when t = 1

Find the value of P when t = 4.

4. a) Find the exact area of the region bounded by the curve with equation

$y = (3 - x)(x + 1)$ $x \in \mathbb{R}$ and the x-axis

b) Find the exact area of the region bounded by the curve with equation

$f(x) = x^2 - 2x + 2$ $x \in \mathbb{R}$ the x-axis , and the lines $x = 1$ and $x = 4$

c) Find the exact area of the region bounded by the curve with equation

$\sqrt{x} + \sqrt{y} = 1$ $x \in \mathbb{R}$, $0 \leq x \leq 1$ and the coordinate axes.

d) Find the exact area of the region bounded by the curve with equation

$y = (x + 1)(x - 2)(x - 4)$ $x \in \mathbb{R}$ and the x-axis.

e) Find the exact area of the region enclosed by the curve with equation

$y = x^2 - 1$ the x-axis, and the lines $x = -2$ and $x = 2$

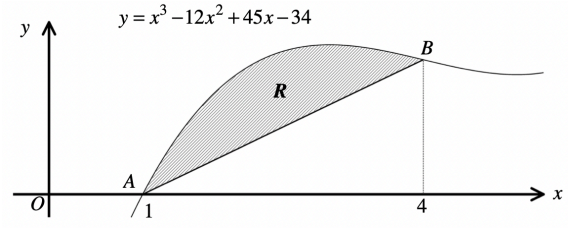
f) Find the exact area of the region enclosed by the curve with equation

$y = x^3 + 3x^2 - 4x$ and the x-axis

5. a) The figure shows the curve with equation $y = x^3 - 12x^2 + 45x - 34$

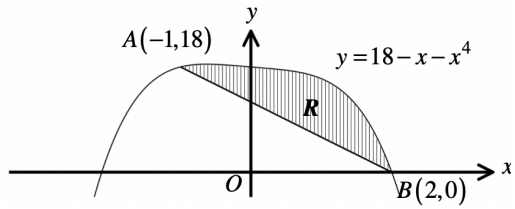
where A has coordinate (1,0) and B has coordinate (4,18)

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



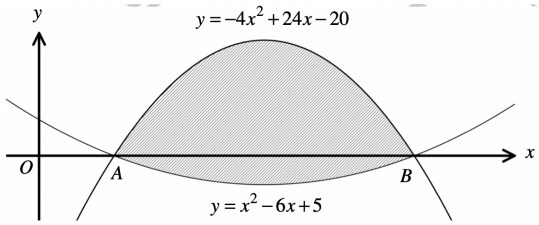
b) The figure shows the curve with equation $y = 18 - x - x^4$

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



c) The figure shows the curves with equations $y = -4x^2 + 24x - 20$ and $y = x^2 - 6x + 5$

Find the area of the shaded region bounded by the two curves.



d) The exact area of the region bounded by the curve with equation $y = x^3$ and $y = mx$ is 6, where m is a positive constant. Find the value of m.

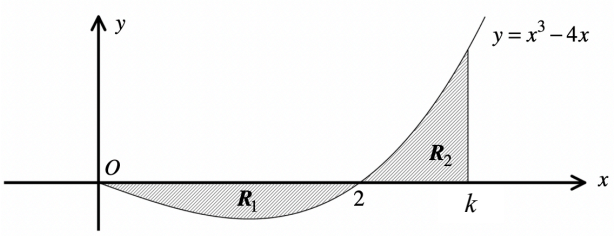
5. e) Find the area of the region bounded by the curve with equation $y = x^3 - x^2$ and the line $y = 2x$ from $x = -1$ to $x = 2$

f) Find the area between the curves with equations $y = p\sqrt{x}$ and $x = p\sqrt{y}$, where p is a positive constant

g) Find the exact area of the region enclosed by the curve with equation $y = |x| - 1$ and the line $y = \frac{1}{2}x$

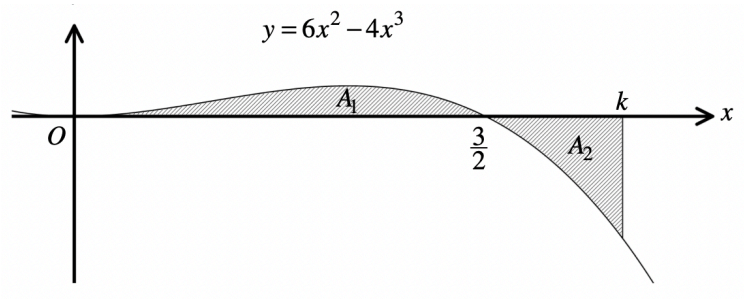
h) The figure shows the cubic curve with equation $y = x^3 - 4x, \quad x \geq 0$

Find the value of k that makes the area of R_1 equal to the area of R_2.



5. i) The figure shows the graph of the curve with equation $y = 6x^2 - 4x^3, \quad x \in \mathbb{R}$

The point $(k\ ,\ 0)$ is such that the area of A_1 equal to the area of A_2. Determine the value of k

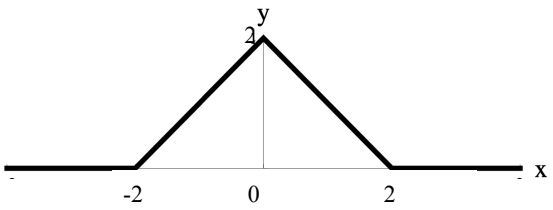


j) Consider the two functions

$f(x) = a - x^2$ and $g(x) = x^4 - a$

Find the values of $a > 0$ for which the area of the region bounded by the x-axis and the curve $y = f(x)$ is bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$.

k) The function $y = f(x)$ given by $f(x) = \begin{cases} 2 - |x| & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$ is shown below



Find the value of $\int_{-1}^{1} f(2x^2 - 2)\ dx$

6a) The polynomial function f(x) is such that f(x) > 0 for all values of x.

Given that $\int_2^4 f(x)\ dx = A$ and $\int_4^6 f(x)\ dx = B$

find the following integrals in terms of A and B

$\int_2^6 f(x)\ dx$ $\int_2^4 f(x)+1\ dx$ $\int_1^3 f(2x)+3\ dx$

$\int_0^2 f(x+2)\ dx$ $\int_{-6}^{-4} f(-x)\ dx$ $\int_{-2}^2 f(x+4)+2\ dx$

$\int_6^8 f(x-2)\ dx$ $\int_2^3 f(2x)\ dx$ $\int_3^7 5f(x-1)\ dx$

b) Match each integral in the first column with an equivalent integral in the 2nd column.

$\int_3^7 f(x)\ dx$	$\int_2^6 f(6-x)\ dx$
$\int_{-1}^3 f(x)\ dx$	$2\int_0^2 f(2x+1)\ dx$
$\int_0^4 f(x)\ dx$	$3\int_0^2 f(3x)\ dx$
$\int_0^6 f(x)\ dx$	$\int_{-3}^1 f(-x)\ dx$
$\int_1^5 f(x)\ dx$	$\int_6^{10} f(x-3)\ dx$

c) Given that $\int_{-1}^3 f(x)\ dx = 12$, find $\int_0^2 f(3-2x)\ dx$

6. d) A continuous function f satisfies $\int_0^4 f(x)\ dx = 10$. Which of the following must be true?

I $\int_0^4 f(4-x)\ dx = 10$

II $\int_0^4 (4-f(x))\ dx = 6$

III $\int_0^{16} f(4x)\ dx = 40$

e) Let f be a strictly increasing and positive continuous function on $\mathbb{R}$.

Arrange the following in increasing order.

$A = \int_0^2 f(x)\ dx$ $B = \int_0^2 f(x-1)\ dx$ $C = \int_0^2 f(6-x)\ dx$

f) Given that $\int_0^4 (f(x))^2\ dx + \int_0^4 f(x)\ dx = \int_0^1 f(x)\ dx$

Are the following statements true or false, or is there insufficient information?

i) $\int_0^4 f(x)\ dx \leq \int_0^1 f(x)\ dx$

ii) $\int_0^1 f(x)\ dx \geq 0$

iii) $f(x) \leq 0$ for some x with $1 \leq x \leq 4$

g) Let $A = \left(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cosx\ dx\right)$ x $\left(\int_{\pi}^{2\pi} sinx\ dx\right)$ x $\left(\int_0^{\frac{\pi}{8}} \frac{1}{cos3x}\ dx\right)$

Which of the following is true:

I A = 0 II A > 0 III A < 0 IV A is not defined

7.

a) Given that $2\int_0^1 f(x)\ dx + 3\int_1^2 f(x)\ dx = 16$ and $\int_0^1 f(x+1)\ dx = 8$

Find the value of $\int_0^2 f(x)\ dx$

b) A curve has equation $f(x) = x(x-a)(x-b)(c-x)$ with $0 < a < b < c$

You are given that $\int_0^c f(x)\ dx = 0$ $\int_0^b f(x)\ dx = -1$ $\int_a^c f(x)\ dx = -4$

Find the total area enclosed by the curve and the x-axis for $0 \le x \le c$

c) A continuous function f satisfies $f(x) + f(6-x) = 2x + 1$.

Find the value of $\int_0^6 f(x)\ dx$

d) It is given that for all real numbers x the function $f(x)$ satisfies:

$$7 + 2f(x) = f(-x) + 4\left(\int_{-1}^1 f(t)\ dt\right)$$

Find the value of $\int_{-1}^1 f(x)\ dx$

7. e) The function $f(x)$ is such that $4f(x)+f(-x) \equiv 5+(2-x^2)\int_{-1}^{1} f(t)\ dt$

Determine the value of $\int_{-1}^{1} f(x)\ dx$

f) The function $f(x)$ satisfies the condition that $f(-x)=f(x)$

You are given that $\int_{-4}^{4} f(x)\ dx = 6$ $\int_{-1}^{4} f(x)\ dx = 2$ $\int_{-5}^{-1} f(x)\ dx = 7$

Find $\int_{0}^{5} f(x)\ dx$

g) The function $f(x)$ is such that $f(0)=0$ and $xf(x)>0$ for $x \neq 0$

You are given that $\int_{-4}^{4} f(x)\ dx = 4$ $\int_{-4}^{4} |f(x)|\ dx = 10$

Find $\int_{-4}^{0} f(|x|)\ dx$

8. a) Use the trapezium rule with five equally spaced ordinates (four strips) to find the value of

$$\int_0^4 \frac{2^x}{x+2}\ dx$$

Use your answer to estimate the value of the following integrals:

$$\int_0^4 \frac{2^x}{x+2} + 3\ dx \qquad \int_0^4 \frac{2^{x+3}}{x+2}\ dx$$

b) A curve has equation $y = f(x) = -x^2 + 8$

State whether the trapezium rule gives an overestimate or underestimate for the total area under the following curves between $x = 0$ and $x = 1$

i) $y = f(x)$

ii) $y = f(x-1)$

iii) $y = 10 - f(x)$

iv) the curve $y = f(x-1)$ reflected in the line $y = 5$

8. c) The function f is such that $0 < f(x) < 1$ for $0 \le x \le 1$

The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ and this produces an underestimate.

If the trapezium rule is used to estimate the following integrals with the same number of equal intervals, does it give an overestimate or underestimate

i) $\int_0^1 3f(x)\ dx$

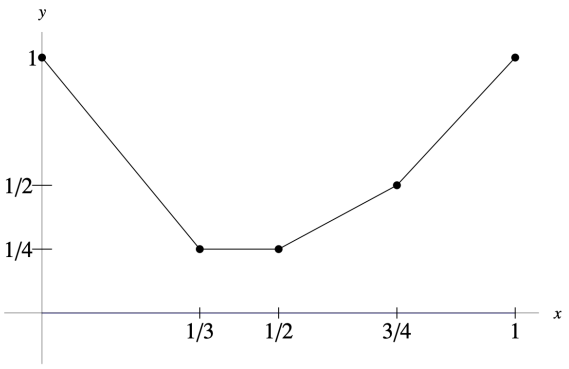
ii) $\int_0^1 f(x)+1\ dx$

iii) $\int_{-1}^0 f(x+1)\ dx$

iv) $\int_{-1}^0 f(-x)\ dx$

v) $\int_0^1 1-f(x)\ dx$

d) The graph of a function f(x), is drawn below for $0 \le x \le 1$



The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ by dividing $0 \le x \le 1$ into n equal intervals

The estimate calculated will equal the actual integral when

I n is a multiple of 4 II n is a multiple of 6

III n is a multiple of 8 IV n is a multiple of 12

e) The trapezium rule is used to estimate $\int_0^1 2^x \ dx$

by dividing the interval $0 \leq x \leq 1$ into N equal subintervals, and the answer achieved is:

I $\frac{1}{2N}\left\{1+\frac{1}{2^{\frac{1}{N}}+1}\right\}$ II) $\frac{1}{2N}\left\{1+\frac{2}{2^{\frac{1}{N}}-1}\right\}$

III) $\frac{1}{N}\left\{1-\frac{1}{2^{\frac{1}{N}}-1}\right\}$ IV) $\frac{1}{2N}\left\{\frac{5}{2^{\frac{1}{N}}+1}-1\right\}$

f) For a function f(x), the trapezium rule with 3 ordinates (2 strips) gives an estimate of **6** for the following definite integral $\int_0^4 f(x) \ dx$.

With 5 ordinates (4 strips) the estimate is **6.2**.

What would the trapezium rule estimate be with 2 ordinates (1 strip) of $\int_1^3 f(x) \ dx$

9. a) Let $F(x) = \int_2^x (t^3 - 4t + 1)\ dt$. Find the gradient of the curve $y = F(x)$ at $x = 2$.

b) Let $F(x) = \int_x^4 (t^2 - 3t - 10)\ dt$. Find the x-coordinates of the stationary points of $F(x)$.

c) Let $F(x) = \int_x^{2x} (t^2 - 7)\ dt$. Find the x-coordinates of the stationary points of $F(x)$.

d) For a positive number a, let $I(a) = \int_0^a (4 - 2^{x^2})\ dx$

Determine the value of a that gives $\frac{dI}{da} = 0$.

e) Let $F(x) = \int_0^x (t - 1)(t - 3)(t - 5)\ dt$.

Determine the values of x for which $F(x)$ is increasing.

9.f) Let $F(x) = \int_2^x (t-1)(t-4)\ dt$

For $1 \leq x \leq 5$, at which value of x does $F(x)$ reach its minimum?

g) Let $F(x) = \int_0^x (t^2-1)(t^2-9)\ dt$

Determine the value(s) of x for which $F(x)$ has its least value.

h) Let $F(x) = \int_0^x (t^2-4)(t^2-9)\ dt.$ Which of the following is the greatest:

$$F(0),\quad F(2),\quad F(3).$$

i) A continuous function f satisfies $\int_1^x f(t)\ dt = x^3 - 3x + 2$ for every real x.

Determine the values of x for which $f(x) = 0$

9. j) A continuous function f satisfies $\int_x^3 f(t)\ dt = x^2 - 6x + 9$ for every real x.

Find $f(x)$.

k) Let $F(x) = \int_0^x (t^2 - 2kt + 3)\ dt$ where k is a real constant.

For which values of k does the curve $y = F(x)$ have exactly two distinct stationary points?

l) A continuous function F satisfies $F'(x) = (x - 1)(x - 4)$.

Let $I_1 = \int_0^1 F'(t)\ dt$ $I_2 = \int_1^4 F'(t)\ dt$ $I_3 = \int_4^5 F'(t)\ dt$

Which of the following must be true?

I $I_1 > 0$ II $I_2 < 0$ III $I_1 < I_3$

10. For these questions we define the following functions:

The floor function $\lfloor t \rfloor$

is the greatest integer less than or equal to *t*: $\lfloor 2.8 \rfloor = 2$ $\lfloor 5 \rfloor = 5$ $\lfloor -0.3 \rfloor = -1$

The ceiling function $\lceil u \rceil$

is the smallest integer greater than or equal to *u:* $\lceil 2.8 \rceil = 3$ $\lceil 5 \rceil = 5$ $\lceil -1.3 \rceil = -1$

The fractional part $\{v\}$

is the fractional part of *v:* $\{1.2\} = 0.2$ $\{-1.6\} = 0.4$ and $\{v\} = v - \lfloor v \rfloor$

a) Start by sketching the graphs of these functions for $-3 \leq x \leq 3$

b) Evaluate $\int_0^4 \lfloor x \rfloor \ dx$

c) Evaluate $\int_{-2}^2 \lceil x \rceil \ dx$

d) Evaluate $\int_0^3 \{x\} \ dx$

10. e) Evaluate $\int_0^3 \lfloor x + \frac{1}{2} \rfloor \ dx$

f) Evaluate $\int_0^3 \lfloor x \rfloor \{x\} \ dx$

g) Evaluate $\int_0^1 \{3x\} \ dx$

h) Evaluate $\int_0^2 \lfloor x^2 \rfloor \ dx$

i) Evaluate $\int_0^{24} 2^{\lceil x \rceil} \ dx$