1) If $f(x)$ satisfies $f(x) = f(x^2)$ for all real x, which of the following must be true:

A $\quad f(4) = f(2)f(2)$

B $\quad f(16) - f(-2) = 0$

C $\quad f(0) = 0$

D $\quad f(-2) + f(4) = 0$

$f(2) = f(2^2) = f(4)$
$f(4) = f(16)$
$f(-2) = f(4)$
$f(0) = f(0)$

Look for counterexamples

2) If $a < 0$ and $b < c$, which of the following must be true?

A $\quad ab < c$ $\quad$ × $\quad a=-1,\ b=-2,\ c=1$

B $\quad ac > b$ $\quad$ × $\quad a=-1,\ b=2,\ c=3$

C $\quad ab > 0$ $\quad$ × $\quad a=-1,\ b=2,\ c=3$

D $\quad ac < 0$ $\quad$ × $\quad a=-1,\ b=-2,\ c=-1$

E $\quad ab > ac$

3) Given that p and q are integers, and that three of the following statements are true, which is the false statement?

A $\quad pq$ is even $\Rightarrow$ at least one of p, q is even

B $\quad p+q$ is even $\Rightarrow$ p, q are both even or both odd

C $\quad 2p+q^2$ is odd $\Rightarrow$ q is odd

D $\quad p^2+2q$ is odd $\Rightarrow$ p is odd

A+B $\Rightarrow$ p, q both even
$\Rightarrow$ C+D not true
So A or B must be false
$\Rightarrow$ C+D are true
$\Rightarrow$ p, q both odd
$\Rightarrow$ A is false

4) Consider the following conjecture:

"If N is a positive integer that consists of the digit 2 followed by an even number of 1 digits, then N is a prime number"

Here are three numbers:

I $N = 21$ (which equals $3x7$) not even number of 1's

II $N = 211$ (a prime number) not counterexample

III $N = 21111$ (which equals 227×93)

Which of these provide(s) a counterexample to the original conjecture?

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

5) Which of the following is a necessary and sufficient condition for

$$\sum_{k=1}^{n} tan(\frac{k\pi}{3}) = 0$$

$\tan\frac{\pi}{3}$ $\tan\frac{2\pi}{3}$ $\tan\pi$ ...

$\sqrt{3}$ $-\sqrt{3}$ 0 $\sqrt{3}$ $-\sqrt{3}$ 0

A n = 3

B n is a multiple of 3

C n is a multiple of 6

D n is a multiple of 3 or n is 1 less than a multiple of 3

E n is a multiple of 6 or n is 1 less than a multiple of 6

multiple of 3
or 1 less than multiple of 3

6) The real numbers *a, b, c, d* satisfy

$0 < a + b < c + d$ and $0 < a + c < b + d$

Which of the following must be true:

Either prove statement is true for all values

Or find a single counterexample to prove false

I $a < d$ ✓ $2a+b+c < 2d+b+c$

II $b < c$ × $a=1$ $b=6$ $c=5$ $d=9$

III $a + b + c + d > 0$ ✓

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

7) The arithmetic mean of five consecutive integers is an odd integer. Does $A \Rightarrow I / II / III$?

A

Which of the following must be true?

I The largest of the integers is even ✗ $n+2$ is odd

II The sum of the integers is odd ✓

III The difference between the largest and smallest of the integers is even. ✓

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

$$\frac{n-2+n-1+n+n+1+n+2}{5} = n$$

n is odd

o e o e o

$o - o = e$

8) Consider the following statement: Need $f(f(x)) = x$ but $f(x) \neq x$

If $f(f(x)) = x$ **then** $f(x) = x$

Which function provides a counterexample:

A $f(x) = 1$ B $f(x) = x$ C $f(x) = \dfrac{1}{x}$ D $f(x) = x^2$

A: $f(f(x)) = 1 \neq x$ ✗

B: ✗

C: $ff(x) = \frac{1}{1/x} = x$ ✓

D: $f(f(x)) = x^4 \neq x$

9) If *a, b* and *c* are integers, consider the statement $\dfrac{ab^2}{c}$ is a positive even integer (*)

Which of the following is a necessary but not sufficient condition for (*) to be true $* \Rightarrow I, II, III$?

I ab is even ✓ II $ab > 0$ III c is even

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

Consider the contrapositive

$* \Rightarrow I$: not $I \Rightarrow$ not $*$

ab odd $\Rightarrow a, b$ odd

$\Rightarrow ab^2$ odd $\Rightarrow c$ odd $\Rightarrow *$ odd

$ab < 0$ $a > 0$ $b < 0$ $ab^2 > 0$

$a = 2$ $b = -1$ $c = 1$

$\frac{ab^2}{c} = 2$ not necessary

$a = 2$ $b = -1$ $c = 1$

$\frac{ab^2}{c} = 2$ not necessary

10) For any real numbers a, b, and c where $a \geq b$, which of these three statements **must** be true?

I $\quad -b \geq -a$ ✓ $\times$ by -1

II $\quad a^2 + b^2 \geq 2ab$ ✓ $(a-b)^2 = a^2 - 2ab + b^2 \geq 0$

III $\quad ac \geq bc$ no if c negative

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

remainder < divisor

11) If x, a and b are positive integers such that when x is divided by a the remainder is b and when x is divided by b the remainder is $(a-2)$, then which of the following must be true?

A $\quad a$ is even

B $\quad b = a - 1$

C $\quad x + b$ is divisible by a

D $\quad a + 2 = b + 1$

$x = pa + b$ $\qquad x = qb + a - 2$

$b < a$ $\qquad a - 2 < b$

$a < b + 2$

$b < a < b + 2$

if a, b are integers there's only one integer between b and b+2

$a = b + 1 \quad b = a - 1$

12) Consider the following statement about positive integers a, b, c and d

If a divides $(b + c)$ and a divides $(c + d)$, then a divides $(b + d)$

Which of the following provide(s) a counterexample to the original conjecture?

I $\quad a = 3, b = 4, c = 5, d = 8$ $\quad a$ not $| \, c + d$

II $\quad a = 5, b = 3, c = 2, d = 8$ ✓

III $\quad a = 2, b = 3, c = 5, d = 7$ example not counterexample

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

13) If $\left|\frac{x}{4}\right| > 1$ which of the following must be true?

A

which of these is necessary for A?

$A \Rightarrow$ I II III

I, II / III is necessary for A

I $\quad x > 4$ $\qquad$ II $\quad x \neq 4$ $\qquad$ III $\quad x < -4$

sufficient $\qquad$ true $\qquad$ sufficient

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

Solving inequality:

$x > 0$, $|x| = x$: $\frac{x}{4} > 1$, $x > 4$

$x < 0$, $|x| = -x$: $-\frac{x}{4} > 1$, $x < -4$

so $x < -4$ or $x > 4$

14) Consider the statement:.

(*) A whole number n is prime if it is 1 less or 5 less than a multiple of 8.

How many counterexamples to (*) are there in the range $0 < n < 50$

A $\quad$ 2

B $\quad$ 3

C $\quad$ 4

D $\quad$ 5

E $\quad$ 6

3 11 19 27 35 43

7 15 23 31 39 47

15) A cubic polynomial is given by $f(x) = x^3 + bx^2 + cx + d$ where b, c and d are constants.

Two of its factors are $(x - 1)$ and $(x + 1)$

Which of the following statements, taken independently, is/are **necessarily** true?

I $\quad$ If $f(0) = k$ then $f(k) = 0$

II $\quad$ $f(x) = x^3 - x$

III $\quad$ The graph of f(x) is symmetrical in the y-axis.

A $\quad$ none of them

B $\quad$ I only

C $\quad$ II only

D $\quad$ III only

E $\quad$ I and II only

F $\quad$ II and III only

G $\quad$ I and III only

H $\quad$ I, II and III

$f(1) = 1 + b + c + d = 0$

$f(-1) = -1 + b - c + d = 0$

$2 + 2c = 0 \quad c = -1$

$b = -d$

$f(x) = x^3 + bx^2 - x - b$

1. $f(0) = k \quad -b = k$

$f(k) = k^3 - k^3 - k + k = 0$ ✓

2. sufficient, but not necessary ✗

$b = 0$

3. $f(-x) = -x^3 + bx^2 + x - b \neq f(x)$ ✗

16) Consider the following statements about the polynomial $p(x)$ where a is a constant

I $\quad p(a) = 0$ and $p'(a) = 0$.

II $\quad p'(a) = 0$ and $p''(a) > 0$

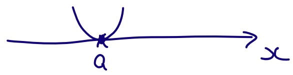
III $\quad p''(a) < 0$ and $p(a) > 0$

Which of these statements are sufficient for $p(x)$ to have a local maximum point at $x = a$?

A none of them
B I only
C II only
D III only
E I and II only
F I and III only
G II and III only
H I, II and III

Does I, II, III $\Rightarrow$ max at $x = a$

I : root + stat point — could be minimum ✗



II : stat. point convex — must be minimum point ✗

III concave, point above x-axis — not necessarily a stat. point ✗

17) Which of the following conditions is sufficient but not necessary for $\dfrac{x}{|x|} < x$

Does I II III $\Rightarrow$ (∗)

I $\quad x > 1$ $\qquad$ II $\quad x > -1$ $\qquad$ III $\quad |x| < 1$

or $-1 < x < 1$

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

I $\quad x > 1 \quad |x| = x$

$\dfrac{x}{|x|} = 1$

$\Rightarrow x > \dfrac{x}{|x|}$ (∗)

(not necessary eg $x = -\frac{1}{2}$) ✓

II Counterexample $x = \frac{1}{2}$

III Counterexample $x = \frac{1}{2}$

Consider $x > 0$
$|x| = x \quad \dfrac{x}{x} < x \quad x > 1$

Consider $x < 0$
$|x| = -x \quad \dfrac{x}{-x} < x \quad -1 < x < 0$

18) Consider the statement: $\quad f(x) > x$ for all real values of $x > 1$

Which one of the following is a negation of this statement?

A $\quad f(x) \le x$ for all real values of $x \le 1$
B $\quad f(x) \le x$ for all real values of $x > 1$
C $\quad f(x) \le x$ for at least one real value of $x \le 1$
D $\quad f(x) \le x$ for at least one real value of $x > 1$
E $\quad f(x) > x$ for at least one real value of $x \le 1$
F $\quad f(x) > x$ for at least one real value of $x > 1$
G $\quad f(x) > x$ for no real values of $x \le 1$
H $\quad f(x) \le x$ for no real values of $x > 1$

Statement

For all $x > 1$ then $f(x) > x$

Negation

There exists $x > 1$ such that NOT $f(x) > x$

There exists $x > 1$ such that $f(x) \le x$

19) A set of cards has a single letter on the front and a single ~~and a~~ number on the back ~~each side~~.

Five of these cards are laid on the table so that only one side of each card is visible.

The cards show the following: Q K 3 6 8

Dan states that all cards with an Q on one side have an even number on the reverse.

Which cards do you need to turn over in order to check his statement? If Q then even

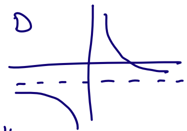
A card Q only
B cards Q and 3
C cards Q and 6 and 8
D cards K and 3
E all of the cards

Q ✓ check for even number
K × irrelevant
3 ✓ check for Q - this could be a contradiction
6 × can be Q or not Q
8 × can be Q or not Q

20) Consider the following statement:

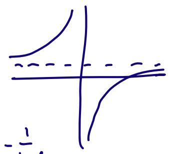
If $f'(x) > 0$ *for all real* x *then* $f(x+1) > f(x)$ *for all real* x

D



$\frac{4}{x} - 1$

E



$\frac{1}{4} - \frac{1}{4x}$

Which function provides a counterexample:

Need $f'(x) > 0$ but $f(x+1) \leq f(x)$ for some x

A $f(x) = 4^x$ B $f(x) = 4x^2 + 1$ C $f(x) = 4x^3$

D $f(x) = \dfrac{4-x}{x}$ E $f(x) = \dfrac{x-1}{4x}$

$f'(x) > 0$ (+ve gradient): A ✓ B × ($x<0$) C × ($x=0$) D × E ✓

$f(x+1) > f(x)$ (strictly increasing) ✓

eg x $-\frac{1}{2}, \frac{1}{2}$

21) x and y are non-zero real numbers. Consider the three statements below:

I $x > y$ **if** $\dfrac{x}{y} > 1$ If $\frac{x}{y} > 1$ then $x > y$

II $\dfrac{x}{y} > 1$ **if and only if** $\dfrac{y}{x} < 1$ $\Rightarrow$ only if yes $\Leftarrow$ if no

III **If** $xy < 1$ **then** both $x < 1$ and $y < 1$

Which of these statements, taken independently, is/are true?

A none of them
B I only
C II only
D III only
E I and II only
F II and III only
G I and III only
H I, II and III

I $x = -4$, $y = -2$ $\frac{x}{y} = 2$ but $x < y$ ×

II $\frac{x}{y} > 1 \Rightarrow x, y$ both +ve or both −ve
$\Downarrow$
$\frac{y}{x} < 1$ ✓
$x < y$
$1 > \frac{y}{x}$

If $\frac{y}{x} < 1$ can have y −ve and x +ve so $\frac{x}{y} \not> 1$ ×

III $x = \frac{1}{12}$, $y = 3$ ×

22) Jill, Kate and Lara each wear a hat. There are three hats: one black, one red, and one blue.

It is known that:

1) If Jill wears black, then Kate wears blue.

2) If Jill wears red, then Lara wears blue.

3) If Kate does not wear red, then Lara wears black.

J	Black	Red	Blue	
K	Blue	Black	Red	Black
L	Red	Blue	Black	Black
	3) x	3) x	✓	3) x

this combination is the only one that works even though no statements apply

What is the colour of the hat Kate is wearing?

A black

B red

C blue

D there is insufficient information to answer the question

Need $f(x) > 0$ for $x \geq 0$ but $f'(x) \leq 0$ for some $x \geq 0$

23) Consider the following statement:

If $f(x) > 0$ for all $x \geq 0$, **then** $f'(x) > 0$ for all $x \geq 0$

A|B $(x \pm \frac{3}{2})^2 + \frac{7}{4} > 0$

Which function provides a counterexample:

A $f(x) = x^2 + 3x + 4$ > 0 B $f(x) = x^2 - 3x + 4$ > 0

C $f(x) = x^2 + 3x - 4$ $f(0) < 0$ D $f(x) = x^2 - 3x - 4$ $f(0) < 0$

	A	B	C	D
$f(x) > 0$	✓	✓	×	×
$f'(x) > 0$	✓	×	✓	×

A: $f'(x) = 2x + 3$ ≥ 0 for $x \geq 0$

B: $f'(x) = 2x - 3$ $f'(0) = -3$

A need $A \Rightarrow$ I, II, III

A is sufficient for I/II/III

I/II/III is necessary for A

24) If $f(x) = \dfrac{x}{x+1}$ for all integers $x \neq -1$, which of the following must be true?

I $f(x+1) > f(x)$ True

II $f(x) > 0$ No when $x = 0$

III $f(x) \neq 0$ No when $x = 0$

None / I only / II only / III only / I and II only / I and III only / II and III only / I II and III

I $f(x+1) = \dfrac{x+1}{x+2}$

Consider $f(x+1) - f(x)$ - is this > 0?

$$= \frac{x+1}{x+2} - \frac{x}{x+1} = \frac{(x+1)^2 - (x^2 + 2x)}{(x+2)(x+1)} = \frac{1}{(x+2)(x+1)} > 0$$

all $x \in \mathbb{Z}$, $x \neq -1, -2$

25) A set *P* of integers is called a *closed set under addition* **if and only if** for any integer *a* in set *P*, *there exists* an integer *k* in *P*, such that *k* is the sum of *a* and *b for all* integers *b which are in P*.

Which of the following is true **if and only if** *P* is **not** *closed under addition*?

A There exists an integer *a* in *P*, such that for any integer *k* in *P*, and any integer *b* in *P*, *k* is not the sum of *a* and *b*.

B There exists an integer *a* in *P*, and an integer *k* in *P*, for which there is no integer *b* in *P*, such that *k* is the sum of *a* and *b*.

C There exists an integer *a* in *P*, such that for any integer *k* in *P*, there is no integer *b* in *P*, such that *k* is the sum of *a* and *b*.

D There exists an integer *a* in *P*, such that for any integer *k* in *P*, there is an integer *b* in *P*, such that *k* is not the sum of *a* and *b*.

E For any integer *a* in *P*, there exists an integer *k* in *P*, and an integer *b* in *P*, such that *k* is not the sum of *a* and *b*.

F For any integer *a* in *P*, there exists an integer *k* in *P*, and an integer *b* in *P*, such that *k* is the sum of *a* and *b*.

G For any integer *a* in *P*, there exists an integer *k* in *P*, such that for any integer *b* in *P*, *k* is not the sum of *a* and *b*.

H For any integer *a* in *P*, and any integer *k* in *P*, there is no integer *b* in *P*, such that *k* is the sum of *a* and *b*.

statement

for any Q there exists R such that S

negation

there exists Q such that not 'there exists R such that S'

—— " —— such that for all R not S

rules out B

↑ rules out EFGH

not S = k is not sum of a and b for some b in P

↑ rules out A, C