1. Integrate the following expressions with respect to x

a) $\int \frac{2+5x}{3x^3}dx$

$= \int \frac{2}{3}x^{-3} + \frac{5}{3}x^{-2}\,dx$

$= -\frac{1}{3}x^{-2} - \frac{5}{3}x^{-1} + C$

$= -\frac{1}{3x^2} - \frac{5}{3x} + C$

b) $\int \frac{\sqrt{x}(4-x)}{2x^2}dx$

$= \int \frac{4x^{1/2} - x^{3/2}}{2x^2}\,dx = \int 2x^{-3/2} - \frac{1}{2}x^{-1/2}\,dx$

$= -4x^{-1/2} - x^{1/2} + C$

$= -\frac{4}{\sqrt{x}} - \sqrt{x} + C$

c) $\int \frac{2x^3 + \sqrt{x^3}}{\sqrt{x}}dx$

$= \int 2x^{5/2} + x\,dx$

$= \frac{4}{7}x^{7/2} + \frac{1}{2}x^2 + C$

d) $\int \frac{(1+\sqrt{x})(3-\sqrt{x})}{x^4}dx$

$= \int \frac{3 + 2x^{1/2} - x}{x^4}\,dx$

$= \int 3x^{-4} + 2x^{-7/2} - x^{-3}\,dx$

$= -x^{-3} - \frac{4}{5}x^{-5/2} + \frac{1}{2}x^{-2} + C$

$= -\frac{1}{x^3} - \frac{4}{5x^{5/2}} + \frac{1}{2x^2} + C$

e) $\int_{-1}^{2} |x-1|(2-x)\,dx$

$|x-1| = x-1 \quad x-1 \geq 0 \quad x \geq 1$

$= 1-x \quad x-1 < 0 \quad x < 1$

so split integral at $x=1$

$= \int_{-1}^{1} (1-x)(2-x)\,dx + \int_{1}^{2} (x-1)(2-x)\,dx$

$= \int_{-1}^{1} x^2 - 3x + 2\,dx + \int_{1}^{2} 3x - 2 - x^2\,dx$

$= \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_{-1}^{1} + \left[\frac{3}{2}x^2 - 2x - \frac{1}{3}x^3\right]_{1}^{2}$

$= \frac{1}{3} - \frac{3}{2} + 2 - \left(-\frac{1}{3} - \frac{3}{2} - 2\right) + \left(6 - 4 - \frac{8}{3}\right) - \left(\frac{3}{2} - 2 - \frac{1}{3}\right)$

$= \frac{5}{6} + \frac{23}{6} - \frac{2}{3} + \frac{5}{6} = \frac{29}{6}$

2. Evaluate the following integrals

a) $\int_1^2 x^4 + 3 - \frac{2}{5x^2}\ dx$

$= \left[\frac{1}{5}x^5 + 3x + \frac{2}{5}x^{-1}\right]_1^2$

$= \left(\frac{32}{5} + 6 + \frac{1}{5}\right) - \left(\frac{1}{5} + 3 + \frac{2}{5}\right)$

$= \frac{30}{5} + 3 = 9$

b) $\int_1^4 \frac{x^3 + 2\sqrt{x}}{x}\ dx$

$= \int_1^4 x^2 + 2x^{-1/2}\ dx = \left[\frac{1}{3}x^3 + 4x^{1/2}\right]_1^4$

$= \left(\frac{64}{3} + 8\right) - \left(\frac{1}{3} + 4\right) = \frac{63}{3} + 4 = 25$

c) $\int_0^4 (x^{1/2} - 3)^2\ dx$

$= \int_0^4 x - 6x^{1/2} + 9\ dx$

$= \left[\frac{1}{2}x^2 - 4x^{3/2} + 9x\right]_0^4$

$= 8 - 32 + 36 = 12$

d) $\int_1^5 3\sqrt{x} - \frac{1}{\sqrt{x}}\ dx$

$= \int_1^5 3x^{1/2} - x^{-1/2}\ dx = \left[2x^{3/2} - 2x^{1/2}\right]_1^5$

$= \left(2\sqrt{125} - 2\sqrt{5}\right) - (2 - 2)$

$= 10\sqrt{5} - 2\sqrt{5} = 8\sqrt{5}$

e) $\int_0^1 \frac{15(2x+1)^2}{2\sqrt{x}}\ dx$

$= \int_0^1 \frac{15(4x^2 + 4x + 1)}{2x^{1/2}}\ dx = \int_0^1 30x^{3/2} + 30x^{1/2} + \frac{15}{2}x^{-1/2}\ dx$

$= \left[12x^{5/2} + 20x^{3/2} + 15x^{1/2}\right]_0^1 = 12 + 20 + 15 - 0$

$= 47$

f) $\int_1^9 6\sqrt{x} - \frac{6}{\sqrt{x}}\ dx$

$= \int_1^9 6x^{1/2} - 6x^{-1/2}\ dx = \left[4x^{3/2} - 12x^{1/2}\right]_1^9$

$= (4 \times 27 - 36) - (4 - 12)$

$= 72 + 8 = 80$

g) $\int_{-2}^3 |x|(1-x)\ dx$

split integral at $x = 0$

$= \int_{-2}^0 -x(1-x)\ dx + \int_0^3 x(1-x)\ dx$

$= \int_{-2}^0 x^2 - x\ dx + \int_0^3 x - x^2\ dx$

$= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-2}^0 + \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^3$

$= 0 - \left(-\frac{8}{3} - 2\right) + \left(\frac{9}{2} - 9\right) - 0$

$= \frac{14}{3} - \frac{9}{2} = \frac{28 - 27}{6} = \frac{1}{6}$

2. h) Given that $\int_3^4 3\sqrt{x} - \frac{4}{\sqrt{x}}\ dx = k\sqrt{3}$ where k is a constant, find k.

$\int_3^4 3x^{1/2} - 4x^{-1/2}\ dx$

$\left[2x^{3/2} - 8x^{1/2}\right]_3^4 = (16 - 16) - (6\sqrt{3} - 8\sqrt{3}) = 2\sqrt{3}$

$k = 2$

i) Given $\int_1^3 f(x)\ dx = \frac{4}{3}$ where $f(x) = 2x^2 + 3x + k$ and k is a constant, find k.

$\int_1^3 2x^2 + 3x + k\ dx = \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + kx\right]_1^3$

$= \left(18 + \frac{27}{2} + 3k\right) - \left(\frac{2}{3} + \frac{3}{2} + k\right)$

$18 + 12 - \frac{2}{3} + 2k = \frac{4}{3}$

$30 + 2k = 2$

$2k = -28$

$k = -14$

j) The positive number a satisfies $\int_0^a 3x^{\frac{7}{2}} + 2x^2 - 12\sqrt{x}\ dx = 0.$ Find a.

$\frac{2}{3}a^{9/2} + \frac{2}{3}a^3 - 8a^{3/2} = 0$

$a^{9/2} + a^3 - 12a^{3/2} = 0$

Let $u = a^{3/2}$

$u^3 + u^2 - 12u = 0$

$u(u+4)(u-3) = 0$

$u > 0 \qquad u = 3$

$a^{3/2} = 3$

$a = 3^{2/3} = \sqrt[3]{9}$

k) Find the smallest value of $I(a) = \int_0^1 (x^2 - a)^2\ dx$ as a varies

$I(a) = \int_0^1 x^4 - 2ax^2 + a^2\ dx$

$= \left[\frac{x^5}{5} - \frac{2a}{3}x^3 + a^2x\right]_0^1$

$= \frac{1}{5} - \frac{2a}{3} + a^2$

$\frac{d}{da} = -\frac{2}{3} + 2a = 0 \qquad a = \frac{1}{3}$

$I\left(\frac{1}{3}\right) = \frac{1}{5} - \frac{2}{9} + \frac{1}{9} = \frac{1}{5} - \frac{1}{9}$

$= \frac{4}{45}$

3. a) Find an equation for y, given $y = 3$ when $x = 1$

$\frac{dy}{dx} = 6x^2 - 4x \qquad x \in \mathbb{R}$

$y = 2x^3 - 2x^2 + c$

$3 = 2 - 2 + c \qquad c = 3$

$y = 2x^3 - 2x^2 + 3$

b) Find an equation for y, given $y = 5$ when $x = 1$

$\frac{dy}{dx} = 4 + \frac{1}{x^2} \qquad x \neq 0$

$= 4 + x^{-2}$

$y = 4x - x^{-1} + c$

$5 = 4 - 1 + c \qquad c = 2$

$y = 4x - \frac{1}{x} + 2$

c) The point (-1 , -1) lies on the curve C whose gradient function is given by

$\frac{dy}{dx} = \frac{5x^3 - 6}{x^3} \qquad x \neq 0$

$= 5 - 6x^{-3}$

$y = 5x + 3x^{-2} + c$

Find an equation for C.

$-1 = -5 + 3 + c \qquad c = 1$

$y = 5x + \frac{3}{x^2} + 1$

d) The points $(0 , -3)$ and $(2 , 7)$ lie on the curve C whose gradient function is given by

$f'(x) = 3x^2 - 4x + k$ where k is a constant. Find an equation for C.

$y = x^3 - 2x^2 + kx + c$

$-3 = c$

$7 = 8 - 8 + 2k - 3$

$2k = 10 \qquad k = 5$

$y = x^3 - 2x^2 + 5x - 3$

e) $f'(x) = 3 - \frac{6}{x^2} \qquad x \neq 0$

Find the value of $f(2)$ given that $2f(1) - f(3) = 3$.

$f(x) = 3x + 6x^{-1} + c$

$f(1) = 9 + c$

$f(3) = 11 + c$

$18 + 2c - 11 - c = 3$

$c = -4$

$f(2) = 6 + 3 - 4 = 5$

f) A quadratic curve C passes through (a , b) and $(2a , 2b)$, where a and b are constants.

The gradient on C is given by $\frac{dy}{dx} = 2x - 6$. Find an equation for C, in terms of a.

$y = x^2 - 6x + c$

$b = a^2 - 6a + c$

$2b = 4a^2 - 12a + c$

$2b = 2a^2 - 12a + 2c$

$0 = 2a^2 - c \qquad c = 2a^2$

$y = x^2 - 6x + 2a^2$

g) It is given that $\frac{dP}{dt} = \frac{15\pi(t-1)}{(1+\sqrt{t})}$ for $t \geq 1$ and $P = 3$ when $t = 1$

Find the value of P when $t = 4$.

$\frac{dP}{dt} = \frac{15\pi(\sqrt{t}-1)(\sqrt{t}+1)}{(1+\sqrt{t})} = 15\pi t^{1/2} - 15\pi$

$P = 10\pi t^{3/2} - 15\pi t + c$

$3 = 10\pi - 15\pi + c \qquad c = 3 + 5\pi$

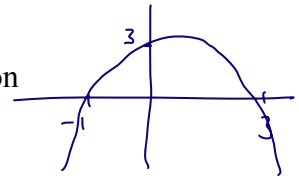
$P = 10\pi(8) - 15\pi(4) + 3 + 5\pi$

$= 80\pi - 60\pi + 5\pi + 3$

$= 25\pi + 3$

4. a) Find the exact area of the region bounded by the curve with equation

$y = (3 - x)(x + 1)$ $x \in \mathbb{R}$ and the x-axis

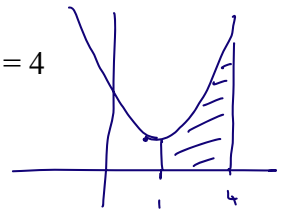


$\int_{-1}^{3} 3 + 2x - x^2 \, dx = \left[3x + x^2 - \frac{1}{3}x^3 \right]_{-1}^{3}$

$= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right) = \frac{32}{3}$

b) Find the exact area of the region bounded by the curve with equation

$f(x) = x^2 - 2x + 2$ $x \in \mathbb{R}$ the x-axis , and the lines $x = 1$ and $x = 4$



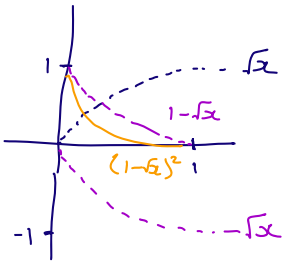
$= (x - 1)^2 + 1$

$\int_{1}^{4} x^2 - 2x + 2 \, dx = \left[\frac{1}{3}x^3 - x^2 + 2x \right]_{1}^{4}$

$= \left(\frac{64}{3} - 16 + 8\right) - \left(\frac{1}{3} - 1 + 2\right) = 12$

c) Find the exact area of the region bounded by the curve with equation

$\sqrt{x} + \sqrt{y} = 1$ $x \in \mathbb{R}$, $0 \le x \le 1$ and the coordinate axes.



$\sqrt{y} = 1 - \sqrt{x}$

$y = (1 - \sqrt{x})^2$

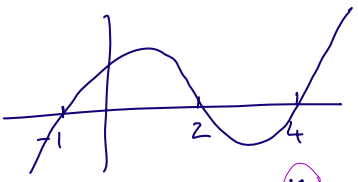
$= 1 - 2\sqrt{x} + x$

$\int_{0}^{1} 1 - 2\sqrt{x} + x \, dx$

$= \left[x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 \right]_{0}^{1} = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}$

d) Find the exact area of the region bounded by the curve with equation

$y = (x + 1)(x - 2)(x - 4)$ $x \in \mathbb{R}$ and the x-axis.



Total Area $= \frac{253}{12}$

$y = (x+1)(x^2 - 6x + 8) = x^3 - 5x^2 + 2x + 8$

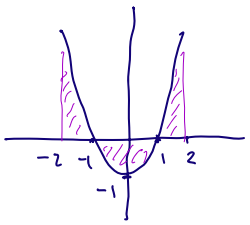
$\int_{-1}^{2} x^3 - 5x^2 + 2x + 8 \, dx = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + x^2 + 8x \right]_{-1}^{2}$

$= \left(4 - \frac{40}{3} + 20\right) - \left(\frac{1}{4} + \frac{5}{3} + 1 - 8\right) = 31 - 15 - \frac{1}{4} = \frac{63}{4}$

$\left[\right]_{2}^{4} = \left(64 - \frac{5}{3} \times 64 + 16 + 32\right) - \left(4 - \frac{40}{3} + 20\right) = 88 + \frac{40}{3} - \frac{320}{3} = \frac{264}{3} - \frac{280}{3} = -\frac{16}{3}$

e) Find the exact area of the region enclosed by the curve with equation

$y = x^2 - 1$ the x-axis, and the lines $x = -2$ and $x = 2$



$\int x^2 - 1 \, dx = \frac{1}{3}x^3 - x$

$\left[\right]_{0}^{1} = \frac{1}{3} - 1 = -\frac{2}{3}$ $\left[\right]_{1}^{2} = \frac{8}{3} - 2 + \frac{2}{3} = \frac{4}{3}$ $\frac{8}{3} - 2 - \left(\frac{1}{3} - 1\right)$

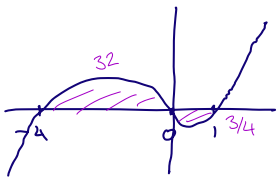
$\left(\frac{4}{3} \times 2\right) + \left(\frac{2}{3} \times 2^2\right)$

$\frac{8}{3} + \frac{4}{3} = 4$

use symmetry:
calculate area for $x > 0$
then double it

f) Find the exact area of the region enclosed by the curve with equation

$y = x^3 + 3x^2 - 4x$ and the x-axis



$= x(x^2 + 3x - 4) = x(x+4)(x-1)$

$\int y \, dx = \frac{1}{4}x^4 + x^3 - 2x^2$

$\left[\right]_{-4}^{0} = -(64 - 64 - 32) = 32$

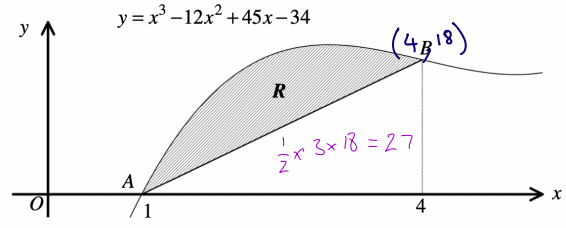
$\left[\right]_{0}^{1} = \frac{1}{4} + 1 - 2 = -\frac{3}{4}$

$32 + \frac{3}{4} = 32\frac{3}{4}$

5. a) The figure shows the curve with equation $y = x^3 - 12x^2 + 45x - 34$

where A has coordinate (1,0) and B has coordinate (4,18)

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



$\frac{1}{2} \times 3 \times 18 = 27$

area between curves

$$\int_1^4 x^3 - 12x^2 + 39x - 28 \, dx$$

$$= \left[\frac{1}{4}x^4 - 4x^3 + \frac{39}{2}x^2 - 28x\right]_1^4$$

$$= \left(64 - 256 + 312 - 112\right) - \left(\frac{1}{4} - 4 + \frac{39}{2} - 28\right)$$

$$= 376 - 368 + 32 - \frac{79}{4}$$

$$= \frac{160}{4} - \frac{79}{4} = \frac{81}{4}$$

or Area below curve − Area of △

$m = 6$ $\quad y - 0 = 6(x-1)$ $\quad y = 6x - 6$

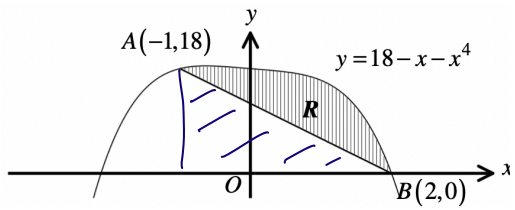
$$\int_1^4 f(x) = \left[\frac{1}{4}x^4 - 4x^3 + \frac{45}{2}x^2 - 34x\right]_1^4$$

$$= \left(64 - 256 + 360 - 136\right) - \left(\frac{1}{4} - 4 + \frac{45}{2} - 34\right) = 32 - -\frac{61}{4} = \frac{189}{4}$$

$A = \frac{189}{4} - 27 = \frac{81}{4}$

b) The figure shows the curve with equation $y = 18 - x - x^4$

Find the area of the finite region R, bounded by the curve and the straight line segment AB.



$\frac{1}{2} \times 3 \times 18 = 27$

$$\int_{-1}^{2} 18 - x - x^4 \, dx = \left[18x - \frac{1}{2}x^2 - \frac{1}{5}x^5\right]_{-1}^{2}$$

$$R = \left(36 - 2 - \frac{32}{5}\right) - \left(-18 - \frac{1}{2} + \frac{1}{5}\right) - 27$$

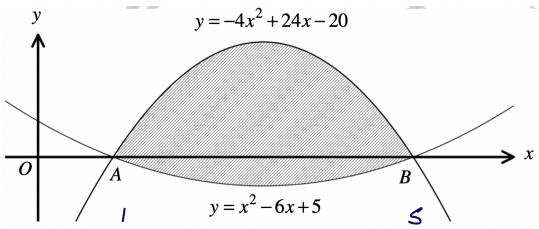
$$= 34 + 18 + \frac{1}{2} - \frac{32}{5} - \frac{1}{5} - 27$$

$$R = 25\frac{1}{2} - \frac{33}{5} = \frac{189}{10}$$

25.5
6.6
18.9

c) The figure shows the curves with equations $y = -4x^2 + 24x - 20$ and $y = x^2 - 6x + 5$

Find the area of the shaded region bounded by the two curves.



$5x^2 - 30x + 25 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

Area between curves:

$$\int_1^5 -5x^2 + 30x - 25 \, dx = \left[-\frac{5}{3}x^3 + 15x^2 - 25x\right]_1^5$$

$$= \left(-\frac{625}{3} + 375 - 125\right) - \left(-\frac{5}{3} + 15 - 25\right)$$

$$= -\frac{620}{3} + 250 + 10 = \frac{780}{3} - \frac{620}{3} = \frac{160}{3}$$

d) The exact area of the region bounded by the curve with equation $y = x^3$ and $y = mx$ is 6,

where m is a positive constant. Find the value of m.

$x^3 = mx$

$x(x^2 - m) = 0$

$x = 0 \quad x = \pm\sqrt{m}$

Let $m > 0$

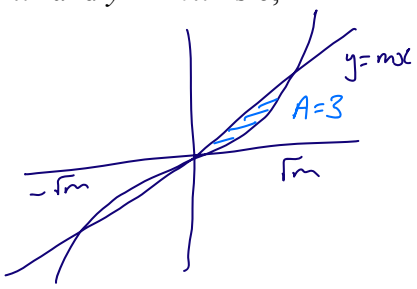
$$\int_0^{\sqrt{m}} mx - x^3 \, dx = 3$$

$$\left[\frac{1}{2}mx^2 - \frac{1}{4}x^4\right]_0^{\sqrt{m}} = 3$$

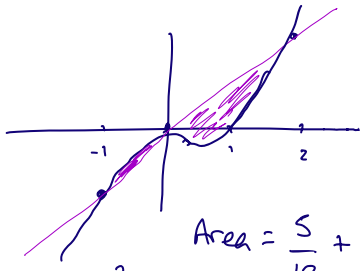
$$\frac{1}{2}m^2 - \frac{1}{4}m^2 = 3$$

$$m^2 = 12$$

$$m = \sqrt{12} = 2\sqrt{3}$$



5. e) Find the area of the region bounded by the curve with equation $y = x^3 - x^2$ and the line $y = 2x$ from $x = -1$ to $x = 2$



$y = x^2(x-1)$

$x^3 - x^2 - 2x = 0$

$x(x^2 - x - 2) = 0$

$x(x-2)(x+1) = 0$

$\int_{-1}^{0} x^3 - x^2 - 2x \, dx$

$\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2\right]_{-1}^{0}$

$-\left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}$

$\int_{0}^{2} 2x + x^2 - x^3 \, dx$

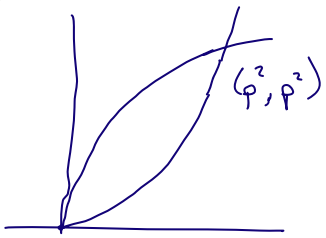
$\left[x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{2}$

$\left(4 + \frac{8}{3} - 4\right) = \frac{8}{3}$

Area $= \frac{5}{12} + \frac{32}{12}$

$= \frac{37}{12}$

f) Find the area between the curves with equations $y = p\sqrt{x}$ and $x = p\sqrt{y}$, where p is a positive constant



$x^2 = p^2 y \quad \frac{x^2}{p^2} = px^{1/2}$

$x^{3/2} = p^3$

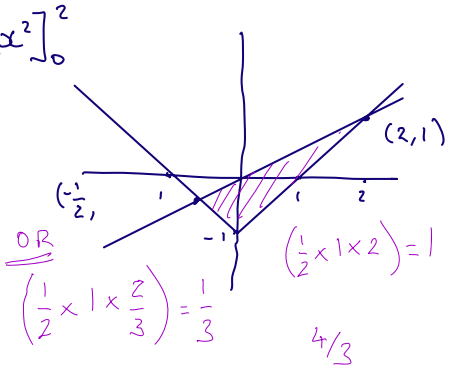
$x = p^2$

$\int_{0}^{p^2} px^{1/2} - \frac{1}{p^2}x^2 \, dx$

$= \left[\frac{2}{3}px^{3/2} - \frac{1}{3p^2}x^3\right]_{0}^{p^2}$

$= \frac{2}{3}p^4 - \frac{1}{3}p^4 = \frac{1}{3}p^4$

g) Find the exact area of the region enclosed by the curve with equation $y = |x| - 1$ and the line $y = \frac{1}{2}x$



$y = |x| \quad y = \frac{1}{2}x + 1$

$x = \frac{1}{2}x + 1 \quad x = 2$

$-x = \frac{1}{2}x + 1 \quad x = -\frac{2}{3}$

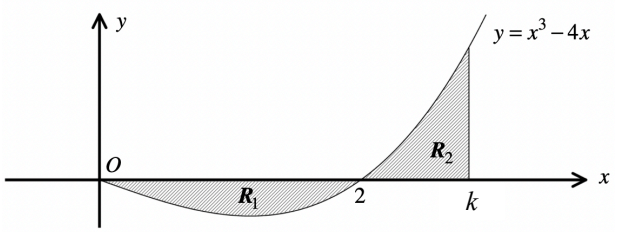
$\int_{-2/3}^{0} \frac{1}{2}x + 1 + x \, dx + \int_{0}^{2} \frac{1}{2}x + 1 - x \, dx$

$\left[\frac{3}{4}x^2 + x\right]_{-2/3}^{0} + \left[x - \frac{1}{4}x^2\right]_{0}^{2}$

$-\left(\frac{1}{3} - \frac{2}{3}\right) + (2 - 1) = \frac{4}{3}$

h) The figure shows the cubic curve with equation $y = x^3 - 4x, \quad x \geq 0$

Find the value of k that makes the area of R_1 equal to the area of R_2.



If $R_1 = R_2 \quad \int_0^k y \, dx = 0$

$\int_{0}^{k} x^3 - 4x \, dx = 0$

$\left[\frac{1}{4}x^4 - 2x^2\right]_{0}^{k} = 0$

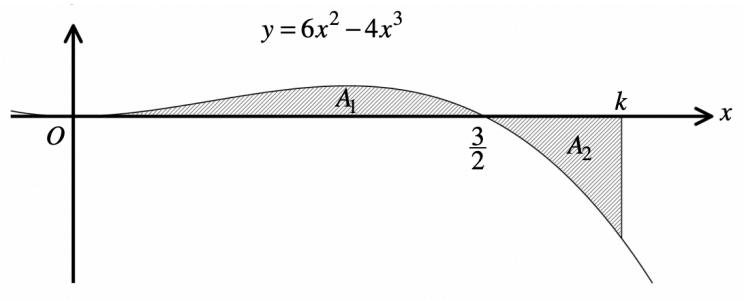
$\frac{1}{4}k^4 - 2k^2 = 0$

$k^2 - 8 = 0$

$k = \sqrt{8}$

5. i) The figure shows the graph of the curve with equation $y = 6x^2 - 4x^3, \quad x \in \mathbb{R}$

The point $(k\ ,\ 0)$ is such that the area of A_1 equal to the area of A_2. Determine the value of k



If $A_1 = A_2$ $\int_0^k y\,dx = 0$

$\int_0^k 6x^2 - 4x^3\,dx = 0$

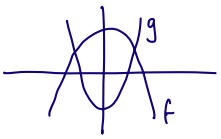
$\left[2x^3 - x^4\right]_0^k = 0$

$2k^3 - k^4 = 0$

$k = 2$

j) Consider the two functions

$f(x) = a - x^2$ and $g(x) = x^4 - a$



Find the values of $a > 0$ for which the area of the region bounded by the x-axis and the curve $y = f(x)$ is bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$.

Need $\int_{-\sqrt{a}}^{\sqrt{a}} a - x^2\,dx > \left|\int_{-\sqrt[4]{a}}^{\sqrt[4]{a}} x^4 - a\,dx\right| \Rightarrow \left[ax - \frac{1}{3}x^3\right]_{-a^{1/2}}^{a^{1/2}} > -\left[\frac{1}{5}x^5 - ax\right]_{-a^{1/4}}^{a^{1/4}}$

or use symmetry and compare $\int_0^{\sqrt{a}} f\,dx$ with $\int_0^{a^{1/4}} -g\,dx$

$\left(a^{3/2} - \frac{1}{3}a^{3/2}\right) - \left(-a^{3/2} + \frac{1}{3}a^{3/2}\right) > -\left(\left(\frac{1}{5}a^{5/4} - a^{5/4}\right) - \left(-\frac{1}{5}a^{5/4} + a^{5/4}\right)\right)$

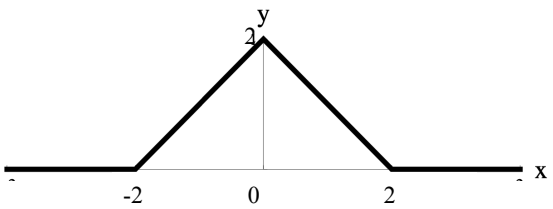
$\frac{2}{3}a^{3/2} + \frac{2}{3}a^{3/2} > \frac{4}{5}a^{5/4} + \frac{4}{5}a^{5/4}$

$\frac{4}{3}a^{3/2} > \frac{8}{5}a^{5/4}$

$a^{1/4} > \frac{24}{20} = \frac{6}{5}$

$a > \left(\frac{6}{5}\right)^4$

k) The function $y = f(x)$ given by $f(x) = \begin{cases} 2 - |x| & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$ is shown below



Find the value of $\int_{-1}^{1} f(2x^2 - 2)\ dx$

$f(x) = x + 2 \quad -2 \le x < 0$

$= 2 - x \quad 0 \le x \le 2$

We want $\int_{-1}^{1} f(2x^2 - 2)\,dx$

for $-1 \le x \le 1$

$0 \le x^2 \le 1$

$0 \le 2x^2 \le 2$

$-2 \le 2x^2 - 2 \le 0$

$\therefore$ use $f(x) = x + 2$

$f(2x^2 - 2) = 2x^2 - 2 + 2$

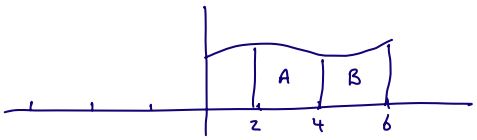
$= 2x^2$

$= \int_{-1}^{1} 2x^2\,dx = \left[\frac{2}{3}x^3\right]_{-1}^{1}$

$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$

6a) The polynomial function f(x) is such that f(x) > 0 for all values of x.

Given that $\int_2^4 f(x)\ dx = A$ and $\int_4^6 f(x)\ dx = B$



find the following integrals in terms of A and B

$\int_2^6 f(x)\ dx$

A + B

$\int_2^4 f(x)+1\ dx$

translation up 1

$A + \square_{2}^{1}$

$A + 2$

$\int_1^3 f(2x)+3\ dx$

$\frac{1}{2}(A+B) + \square_{2}^{3}$

$\frac{1}{2}(A+B) + 6$

$\int_0^2 f(x+2)\ dx$

translation 2 left

A

$\int_{-6}^{-4} f(-x)\ dx$

reflection in y-axis

B

$\int_{-2}^{2} f(x+4)+2\ dx$

$A+B + \square_{4}^{2}$

$A+B+8$

$\int_6^8 f(x-2)\ dx$

translation 2 right

B

$\int_2^3 f(2x)\ dx$

enlargement SF $\frac{1}{2}$

$\frac{1}{2}B$

$\int_3^7 5f(x-1)\ dx$

$5(A+B)$

b) Match each integral in the first column with an equivalent integral in the 2nd column.

$\int_3^7 f(x)\ dx$	$\int_2^6 f(6-x)\ dx$
$\int_{-1}^3 f(x)\ dx$	$2\int_0^2 f(2x+1)\ dx$
$\int_0^4 f(x)\ dx$	$3\int_0^2 f(3x)\ dx$
$\int_0^6 f(x)\ dx$	$\int_{-3}^1 f(-x)\ dx$
$\int_1^5 f(x)\ dx$	$\int_6^{10} f(x-3)\ dx$

reflection in $x=3$ $= \int_0^4 f(x)\ dx$

$f(x) \to f(x+1) \to f(2x+1)$

$\int_1^5$ ← translation $\int_0^4$ ← stretch $\int_0^2$

stretch SF $\frac{1}{3}$ ×3 preserves area

reflection in y-axis $= \int_{-1}^3$

translation 3 right

c) Given that $\int_{-1}^3 f(x)\ dx = 12$, find $\int_0^2 f(3-2x)\ dx$

$f(x) \to f(x+3) \to f(2x+3) \to f(-2x+3)$

translate 3 left; stretch SF $\frac{1}{2}$; reflect in y-axis

$\int_{-1}^3 = 12$ $\int_{-4}^0 = 12$ $\int_{-2}^0 = 6$ $\int_0^2 f(3-2x)\ dx = 6$

6. d) A continuous function f satisfies $\int_0^4 f(x)\ dx = 10$. Which of the following must be true?

I ✓ $\int_0^4 f(4-x)\ dx = 10$ reflection in $x = 2$ $\int_0^4 f(x)\,dx = \int_0^4 f(4-x)\,dx$ so TRUE

II ✓ $\int_0^4 (4-f(x))\ dx = 6$ $= \int_0^4 4\,dx - 10 = 16 - 10 = 6$ so TRUE

III ✗ $\int_0^{16} f(4x)\ dx = 40$ stretch SF $\frac{1}{4}$ so $\int_0^1 f(4x)\,dx = \frac{10}{4} = \frac{5}{2}$ is only conclusion

e) Let f be a strictly increasing and positive continuous function on $\mathbb{R}$.

Arrange the following in increasing order.

$A = \int_0^2 f(x)\ dx$ $B = \int_0^2 f(x-1)\ dx$ translation 1 right $= \int_{-1}^1 f(x)\,dx$ $C = \int_0^2 f(6-x)\ dx$ reflection in $x=3$ $= \int_4^6 f(x)\,dx$

As f is increasing and integral widths are identical $B < A < C$

f) Given that $\int_0^4 (f(x))^2\ dx + \int_0^4 f(x)\ dx = \int_0^1 f(x)\ dx \Rightarrow$ (+ve)

Are the following statements true or false, or is there insufficient information?

i) $\int_0^4 f(x)\ dx \le \int_0^1 f(x)\ dx$ True $\int_0^1 f(x)\,dx - \int_0^4 f(x)\,dx = \int_0^4 (f(x))^2\,dx \ge 0$

ii) $\int_0^1 f(x)\ dx \ge 0$ Insufficient could be negative if $\int_0^4 f(x)\,dx < 0$

iii) $f(x) \le 0$ for some x with $1 \le x \le 4$ True $\int_0^4 (f(x))^2\,dx + \int_1^4 f(x)\,dx = 0$

g) Let $A = \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx\right) \times \left(\int_{\pi}^{2\pi} \sin x\ dx\right) \times \left(\int_0^{\frac{\pi}{8}} \frac{1}{\cos 3x}\ dx\right)$ (>0) (<0) (>0)

consider the graphs for the limits given and where they are positive or negative

$\cos x \ge 0$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ so integral > 0

+ve × −ve × +ve = −ve

Which of the following is true:

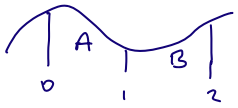
I $A = 0$ II $A > 0$ III $A < 0$ IV A is not defined

7.

a) Given that $2\int_0^1 f(x)\ dx + 3\int_1^2 f(x)\ dx = 16$ and $\int_0^1 f(x+1)\ dx = 8$

Let $A = \int_0^1 f(x)\,dx$ and $B = \int_1^2 f(x)\,dx$ $\quad B = 8$

Find the value of $\int_0^2 f(x)\ dx$



$2A + 3B = 16$

$2A + 24 = 16 \quad A = -4$

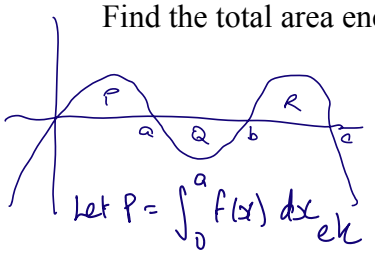
$\int_0^2 f(x)\,dx = A + B$

$= 4$

b) A curve has equation $f(x) = x(x-a)(x-b)(c-x)$ with $0 < a < b < c$

You are given that $\int_0^c f(x)\ dx = 0 \quad \int_0^b f(x)\ dx = -1 \quad \int_a^c f(x)\ dx = -4$

Find the total area enclosed by the curve and the x-axis for $0 \le x \le c$



Let $P = \int_0^a f(x)\,dx$ etc

$P + Q + R = 0 \quad R = 1$

$P + Q = -1 \quad Q = -5$

$Q + R = -4 \quad P = 4$

Area $= 4 + 5 + 1$

$= 10$

c) A continuous function f satisfies $f(x) + f(6-x) = 2x + 1$.

integrate both sides

Find the value of $\int_0^6 f(x)\ dx$

$\int_0^6 f(x)\,dx + \int_0^6 f(6-x)\,dx = \int_0^6 2x+1\,dx$

for $0 \le x \le 6$

$\int f(x) = \int f(6-x)$

(reflection in $x = 3$)

$2\int_0^6 f(x)\,dx = \left[x^2 + x\right]_0^6 = 42$

$\int_0^6 f(x)\,dx = 21$

d) It is given that for all real numbers x the function $f(x)$ satisfies:

$$7 + 2f(x) = f(-x) + 4\left(\int_{-1}^{1} f(t)\ dt\right)$$

Let $\int_{-1}^{1} f(t)\,dt = k$

+ integrate both sides

Find the value of $\int_{-1}^{1} f(x)\ dx = k$

$\int_{-1}^{1} 7 + 2f(x)\,dx = \int_{-1}^{1} f(-x) + 4k\,dx$

$\left[7x\right]_{-1}^{1} + 2k = k + \left[4kx\right]_{-1}^{1}$

$14 + 2k = k + 8k$

$14 = 7k$

$k = 2$

7. e) The function $f(x)$ is such that $4f(x)+f(-x) \equiv 5+(2-x^2)\int_{-1}^{1} f(t)\ dt$

Determine the value of $\int_{-1}^{1} f(x)\ dx$ $= k$

$4f(x) + f(-x) \equiv 5 + 2k - kx^2$

$\int_{-1}^{1} 4f(x) + f(-x)\ dx = \int_{-1}^{1} 5 + 2k - kx^2\ dx$

Note $\int_{-1}^{1} f(x)\ dx = \int_{-1}^{1} f(-x)\ dx = k$

$5k = \left[5x + 2kx - \frac{1}{3}kx^3\right]_{-1}^{1}$

$5k = 5 + 2k - \frac{1}{3}k - \left(-5 - 2k + \frac{1}{3}k\right)$

$= 10 + \frac{10}{3}k$

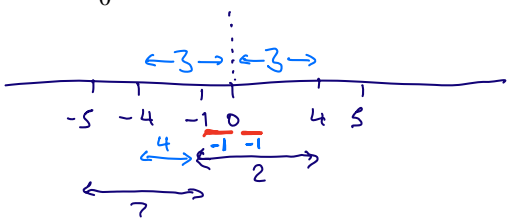
$\frac{5}{3}k = 10$

$k = 6$

f) The function $f(x)$ satisfies the condition that $f(-x) = f(x)$ (even function)

You are given that $\int_{-4}^{4} f(x)\ dx = 6$ $\int_{-1}^{4} f(x)\ dx = 2$ $\int_{-5}^{-1} f(x)\ dx = 7$

Find $\int_{0}^{5} f(x)\ dx$



$\int_0^5 f(x)\ dx = \int_0^1 f(x)\ dx + \int_1^5 f(x)\ dx$

$= -1 + 7$

$= 6$

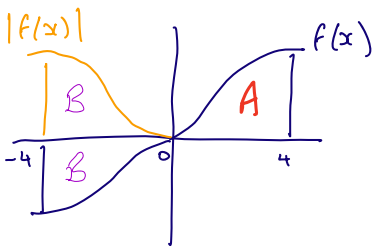
g) The function $f(x)$ is such that $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$

You are given that $\int_{-4}^{4} f(x)\ dx = 4$ $A - B = 4$

$\int_{-4}^{4} |f(x)|\ dx = 10$ $A + B = 10$

Find $\int_{-4}^{0} f(|x|)\ dx$ $A = 7$ $B = 3$

Alternatively Let areas = A, B rather than integrals so A, B ≥ 0



You can choose to define your A, B etc as areas or integrals - just be clear which method you are using or you might have some sign confusion.

8. a) Use the trapezium rule with five equally spaced ordinates (four strips) to find the value of

$$\int_0^4 \frac{2^x}{x+2}\,dx$$

0	1	2	3	4
$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{8}{5}$	$\frac{8}{3}$

$$\frac{1}{2}\left(\frac{1}{2}+\frac{8}{3}+2\left(\frac{2}{3}+1+\frac{8}{5}\right)\right) = \frac{19}{12}+\frac{20}{12}+\frac{8}{5} = \frac{13}{4}+\frac{8}{5}$$

$$= \frac{65+32}{20} = \frac{97}{20} = 4\frac{17}{20}$$

Use your answer to estimate the value of the following integrals:

$$\int_0^4 \frac{2^x}{x+2}+3\,dx$$

$$\frac{97}{20}+12 = 16\frac{17}{20}$$

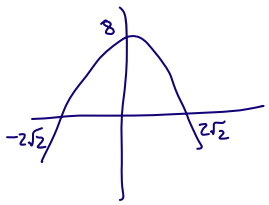
$$\int_0^4 \frac{2^{x+3}}{x+2}\,dx = \int_0^4 \frac{2^3\,2^x}{x+2}\,dx$$

$$= 8\times\frac{97}{20} = \frac{194}{5}$$

b) A curve has equation $y = f(x) = -x^2 + 8$

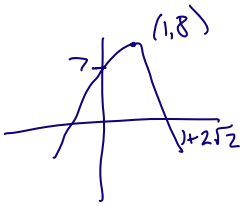
State whether the trapezium rule gives an overestimate or underestimate for the total area under the following curves between $x = 0$ and $x = 1$

i) $y = f(x)$



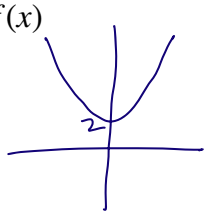
underestimate

ii) $y = f(x-1)$



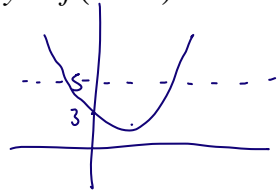
underestimate

iii) $y = 10 - f(x)$



overestimate

iv) the curve $y = f(x-1)$ reflected in the line $y = 5$



overestimate

8. c) The function f is such that $0 < f(x) < 1$ for $0 \le x \le 1$

The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ and this produces an underestimate.

If the trapezium rule is used to estimate the following integrals with the same number of equal intervals, does it give an overestimate or underestimate

i) $\int_0^1 3f(x)\ dx$ under (stretch has no effect)

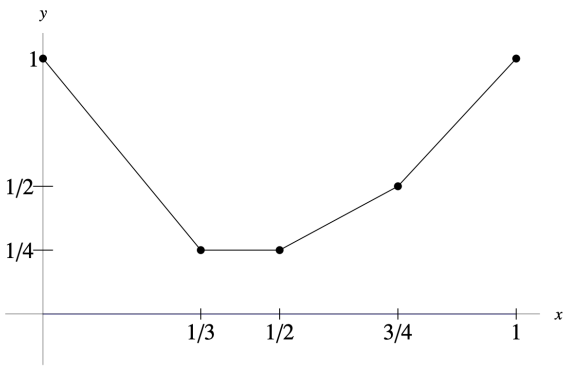
ii) $\int_0^1 f(x)+1\ dx$ under (translation has no effect)

iii) $\int_{-1}^0 f(x+1)\ dx$ under (translation has no effect)

iv) $\int_{-1}^0 f(-x)\ dx$ under (reflection in y-axis has no effect)

v) $\int_0^1 1-f(x)\ dx$ over (reflection in x-axis swaps over $\longleftrightarrow$ under)

d) The graph of a function f(x), is drawn below for $0 \le x \le 1$



For the estimate to equal the actual integral, all the trapezia must lie exactly on f(x) (not over or under)

eg 4 equal intervals would have $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and the trapezium between $\frac{1}{4}$ and $\frac{1}{2}$ would not lie exactly on f(x) ...

The trapezium rule is used to estimate $\int_0^1 f(x)\ dx$ by dividing $0 \le x \le 1$ into n equal intervals

The estimate calculated will equal the actual integral when

I n is a multiple of 4 II n is a multiple of 6

III n is a multiple of 8 IV n is a multiple of 12

e) The trapezium rule is used to estimate $\int_0^1 2^x \; dx$

x	0	$\frac{1}{N}$	$\frac{2}{N}$	$\frac{3}{N}$	...	$\frac{N-1}{N}$	1
2^x	1	$2^{\frac{1}{N}}$	$2^{\frac{2}{N}}$	..			2

by dividing the interval $0 \le x \le 1$ into N equal subintervals, and the answer achieved is:

I $\frac{1}{2N}\left\{1+\frac{1}{2^{\frac{1}{N}}+1}\right\}$ II) $\frac{1}{2N}\left\{1+\frac{2}{2^{\frac{1}{N}}-1}\right\}$

III) $\frac{1}{N}\left\{1-\frac{1}{2^{\frac{1}{N}}-1}\right\}$ IV) $\frac{1}{2N}\left\{\frac{5}{2^{\frac{1}{N}}+1}-1\right\}$

G.P. $a = 1$, $r = 2^{\frac{1}{N}}$

$S_n = \frac{1-2^{\frac{1}{N}\cdot N}}{1-2^{\frac{1}{N}}} = \frac{1}{2^{\frac{1}{N}}-1}$

$= \frac{1}{2N}\left\{1+2+2\left(2^{\frac{1}{N}}+2^{\frac{2}{N}}+\ldots 2^{\frac{N-1}{N}}\right)\right\}$

$= \frac{1}{2N}\left\{1+2\left(1+2^{\frac{1}{N}}+..+2^{\frac{N-1}{N}}\right)\right\} = \frac{1}{2N}\left\{1+\frac{2}{2^{\frac{1}{N}}-1}\right\}$

f) For a function f(x), the trapezium rule with 3 ordinates (2 strips) gives an estimate of **6** for the following definite integral $\int_0^4 f(x) \; dx$.

With 5 ordinates (4 strips) the estimate is **6.2**.

What would the trapezium rule estimate be with 2 ordinates (1 strip) of $\int_1^3 f(x) \; dx$

2 strips: $\frac{1}{2}\times 2\left(f(0)+f(2)+2f(2)\right) = 6$ (1)

4 strips: $\frac{1}{2}\left(f(0)+f(4)+2\left(f(1)+f(2)+f(3)\right)\right) = 6.2$ (2)

Sub (1) in (2) $6 + 2\left(f(1)+f(3)\right) = 12.4$

1 strip: $\frac{1}{2}\times 2\left(f(1)+f(3)\right) = \frac{1}{2}\times 6.4 = 3.2$

From the Fundamental Theorem of Calculus $F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$

9. a) Let $F(x) = \int_2^x (t^3 - 4t + 1)\,dt$. Find the gradient of the curve $y = F(x)$ at $x = 2$.

From FTC

$F'(x) = x^3 - 4x + 1$

You could also integrate, then differentiate to get the same result!

$= \left[\frac{1}{4}t^4 - 2t^2 + t\right]_2^x = \frac{1}{4}x^4 - 2x^2 + x - (4 - 8 + 2)$

$F'(x) = x^3 - 4x + 1$ $\qquad F'(2) = 1$

b) Let $F(x) = \int_x^4 (t^2 - 3t - 10)\,dt$. Find the x-coordinates of the stationary points of $F(x)$.

$F(x) = \int_4^x -t^2 + 3t + 10\,dt$

$F'(x) = -x^2 + 3x + 10 = 0$

$x^2 - 3x - 10 = 0$

$(x - 5)(x + 2)$

$x = -2, 5$

c) Let $F(x) = \int_x^{2x} (t^2 - 7)\,dt$. Find the x-coordinates of the stationary points of $F(x)$.

$F(x) = \int_x^{2x} t^2 - 7\,dt = \int_a^{2x} t^2 - 7\,dt - \int_a^x t^2 - 7\,dt$

$= 2(4x^2 - 7) - (x^2 - 7)$

$= 7x^2 - 7$

$F'(x) = 7x^2 - 7 = 0$

$x = \pm 1$

d) For a positive number a, let $I(a) = \int_0^a (4 - 2^{x^2})\,dx$

Determine the value of a that gives $\frac{dI}{da} = 0$.

FTC: $\frac{dI}{da} = 4 - 2^{a^2}$

$4 - 2^{a^2} = 0$

$2^{a^2} = 2^2$

$a^2 = 2$

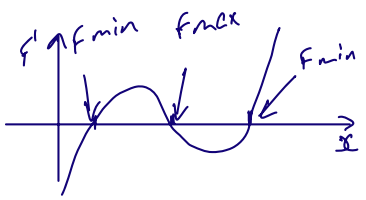
$a = \sqrt{2}$

e) Let $F(x) = \int_0^x (t - 1)(t - 3)(t - 5)\,dt$.

Determine the values of x for which $F(x)$ is increasing.

$F'(x) = (x - 1)(x - 3)(x - 5)$

$F'(x) = 0$ $\quad x = 1, 3, 5$



$F(x)$ increasing

$F'(x) > 0$ $\quad 1 < x < 3,\ x > 5$

Alternatively integrate and differentiate... (much more work!)

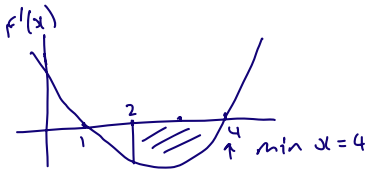
$F(x) = \int_2^x t^2 - 5t + 4 \, dt = \left[\frac{1}{3}t^3 - \frac{5}{2}t^2 + 4t\right]_2^x$

$= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x - \frac{8}{3} + 10 - 8$

9.f) Let $F(x) = \int_2^x (t-1)(t-4) \, dt$

For $1 \le x \le 5$, at which value of x does $F(x)$ reach its minimum?

$F'(x) = (x-1)(x-4)$



F represents the signed area from $x = 2$. This is negative as it is below the x-axis and becomes more negative until $x = 4$. Therefore min F is at $x = 4$

$F'(x) = x^2 - 5x + 4 = 0$

$x = 1, 4$

$F''(x) = 2x - 5$

∴ min at $x = 4$ max at $x = 1$

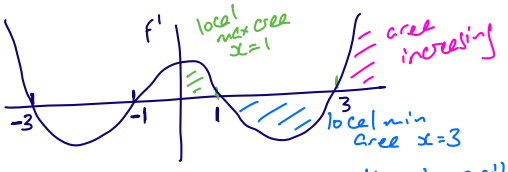
g) Let $F(x) = \int_0^x (t^2-1)(t^2-9) \, dt$

Determine the value(s) of x for which $F(x)$ has its least value.

Stat points: $F'(x) = (x^2-1)(x^2-9) = 0$

$x = \pm 1 \pm 3$

min $F(x)$ at $x = -1, 3$



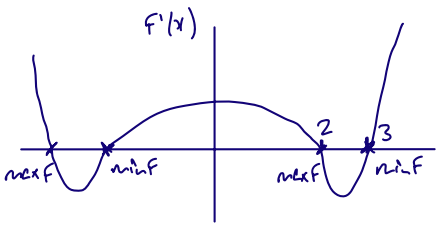
As F' changes from negative to positive at $x = 3$ F has a minimum point

h) Let $F(x) = \int_0^x (t^2-4)(t^2-9) \, dt$. Which of the following is the greatest:

$F(0), \quad F(2), \quad F(3).$

$F(x) = \int_0^x (t^2-4)(t^2-9) \, dt$

$F'(x) = 0 \quad x = \pm 2 \pm 3$



$F(0) = 0$

$F(2) = \max F > 0$

$F(3) = \min F$

greatest = $F(2)$

i) A continuous function f satisfies $\int_1^x f(t) \, dt = x^3 - 3x + 2$ for every real x.

Determine the values of x for which $f(x) = 0$

$\int_1^x f(t) \, dt = x^3 - 3x + 2$

$f(x) = 3x^2 - 3 = 0 \qquad x = \pm 1$

9. j) A continuous function f satisfies $\int_x^3 f(t)\ dt = x^2 - 6x + 9$ for every real x.

Find $f(x)$.

$\int_3^x f(t)\,dt = -x^2 + 6x - 9$

$f(x) = -2x + 6$

k) Let $F(x) = \int_0^x (t^2 - 2kt + 3)\ dt$ where k is a real constant.

For which values of k does the curve $y = F(x)$ have exactly two distinct stationary points?

$F'(x) = x^2 - 2kx + 3$

2 stat points $\Delta > 0$

$4k^2 - 4(3) > 0$

$k^2 > 3$ $\quad k < -\sqrt{3},\ k > \sqrt{3}$

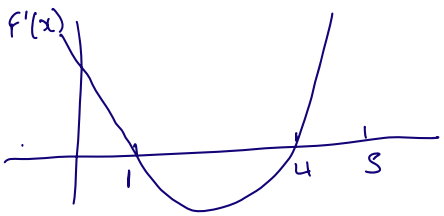
l) A continuous function F satisfies $F'(x) = (x-1)(x-4)$.

Let $I_1 = \int_0^1 F'(t)\ dt$ $\quad I_2 = \int_1^4 F'(t)\ dt$ $\quad I_3 = \int_4^5 F'(t)\ dt$

Which of the following must be true?

I $I_1 > 0$ $\quad$ II $I_2 < 0$ $\quad$ III $I_1 < I_3$

$I_1 = \int_0^1 F'(t)\,dt = F(1) - F(0)$ $\quad I_2 = \int_1^4 = F(4) - F(1)$ $\quad I_3 = \int_4^5 = F(5) - F(4)$



By checking the sign of $F'(x)$ we can see

$F(x)$ is increasing for $x < 1$

decreasing for $1 < x < 4$

increasing for $x > 4$

with max F at $x = 1$, min F at $x = 4$

Therefore $F(0) < F(1)$ $\quad I_1 > 0$ ✓

$F(4) < F(1)$ $\quad I_2 < 0$ ✓

$F(4) < F(5)$ $\quad I_3 > 0$

in fact by symmetry $I_3 = I_1$

so only I and II are true

10. For these questions we define the following functions:

The floor function $\lfloor t \rfloor$

is the greatest integer less than or equal to t: $\lfloor 2.8 \rfloor = 2$ $\lfloor 5 \rfloor = 5$ $\lfloor -0.3 \rfloor = -1$

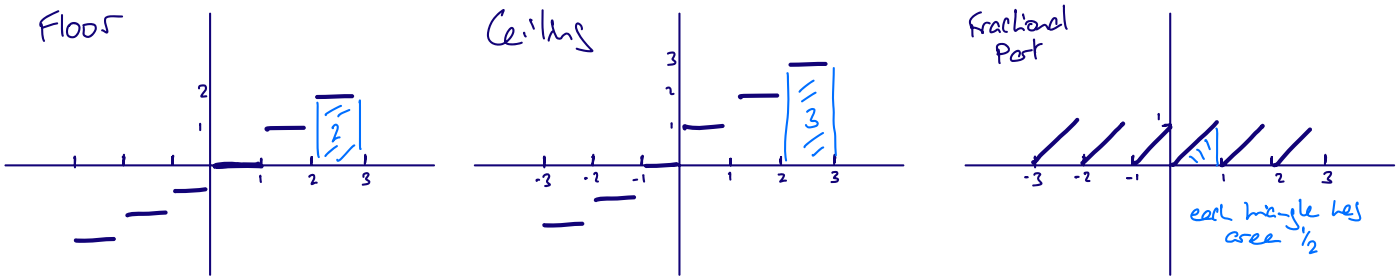
The ceiling function $\lceil u \rceil$

is the smallest integer greater than or equal to u: $\lceil 2.8 \rceil = 3$ $\lceil 5 \rceil = 5$ $\lceil -1.3 \rceil = -1$

The fractional part $\{v\}$

is the fractional part of v: $\{1.2\} = 0.2$ $\{-1.6\} = 0.4$ and $\{v\} = v - \lfloor v \rfloor$

a) Start by sketching the graphs of these functions for $-3 \le x \le 3$



b) Evaluate $\int_0^4 \lfloor x \rfloor \, dx$

$= \int_0^1 0 \, dx + \int_1^2 1 \, dx + \int_2^3 2 \, dx + \int_3^4 3 \, dx$

$= 0 + 1 + 2 + 3$

$= 6$

each of these is the area of the rectangle between the limits

c) Evaluate $\int_{-2}^{2} \lceil x \rceil \, dx$

$= \int_{-2}^{-1} (-1) \, dx + \int_{-1}^{0} 0 \, dx + \int_0^1 1 \, dx + \int_1^2 2 \, dx$

$= -1 + 0 + 1 + 2$

$= 2$

d) Evaluate $\int_0^3 \{x\} \, dx$

$= \int_0^1 x \, dx + \int_1^2 x - 1 \, dx + \int_2^3 x - 2 \, dx$

$= \left[\frac{1}{2}x^2\right]_0^3 - [x]_1^2 - [2x]_2^3$

$= \frac{9}{2} - 1 - (6 - 4) = \frac{3}{2}$

or Area of 3 triangles $= 3 \times \frac{1}{2} = \frac{3}{2}$

10. e) Evaluate $\int_0^3 \left\lfloor x + \frac{1}{2} \right\rfloor \, dx$

$= \int_0^{1/2} 0 \, dx + \int_{1/2}^{3/2} 1 + \int_{3/2}^{5/2} 2 + \int_{5/2}^{3} 3$

$= 0 + 1 + 2 + 3/2 = 9/2$

f) Evaluate $\int_0^3 \lfloor x \rfloor \{x\} \, dx = \int_0^1 0 \, dx \int_1^2 (1)(x-1) \, dx + \int_2^3 2(x-2) \, dx$

$= \left[\frac{1}{2}x^2 - x\right]_1^2 + \left[x^2 - 4x\right]_2^3$

$= \frac{1}{2} + 9 - 12 - 4 + 8 = 3/2$

g) Evaluate $\int_0^1 \{3x\} \, dx = \int_0^{1/3} 3x \, dx + \int_{1/3}^{2/3} 3x - 1 \, dx + \int_{2/3}^{1} 3x - 2 \, dx$

$= \int_0^1 3x \, dx - \left[x\right]_{1/3}^{2/3} - \left[2x\right]_{2/3}^{1}$

$= \frac{3}{2} - \left(\frac{1}{3}\right) - \left(2 - \frac{4}{3}\right) = \frac{3}{2} - \frac{1}{3} - \frac{2}{3} = \frac{1}{2}$

* or recognise this as a stretch SF $\frac{1}{3}$ so area $= \frac{3}{2} \times \frac{1}{3} = \frac{1}{2}$ [using area from graph in a)]

h) Evaluate $\int_0^2 \lfloor x^2 \rfloor \, dx = \int_0^1 0 \, dx + \int_1^{\sqrt{2}} 1 \, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 \, dx + \int_{\sqrt{3}}^{2} 3 \, dx$

$= 0 + \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$

$= 5 - \sqrt{2} - \sqrt{3}$

i) Evaluate $\int_0^{24} 2^{\lceil x \rceil} \, dx = \int_0^1 2 + \int_1^2 2^2 + \int_2^3 2^3 + \ldots + \int_{23}^{24} 2^{24}$

$= 2 + 4 + 8 + \ldots 2^{24}$

Geometric series

$a = 2 \quad r = 2$

$S_{24} = 2(2^{24} - 1)$

$= 2^{25} - 2$