1. The function f is defined such that $3f(x) - f(-x) = 8x + 10$
find the value of $f(2)$

A 5
B 6
C 7
D 8
E 9

We want $f(2)$ so substitute $x = 2$
① $3f(2) - f(-2) = 26$
We are not told that f is even/odd so try $x = -2$
② $3f(-2) - f(2) = -6$
These now look like simultaneous equations ① ×3 + ②
$8f(2) = 72$
$f(2) = 9$

2. The function f satisfies $2f(x) + f\left(\frac{x+5}{x-1}\right) = 3x, \quad x \in \mathbb{R}$
Find the value of $f(3)$

A 2
B 3
C 4
D 5
E 6

Try $x = 3$
$2f(3) + f(4) = 9$ ①
Try $x = 4$
$2f(4) + f(3) = 12$ ②

Again we have simultaneous equations
$2\times$① − ② $\quad 3f(3) = 6$
$f(3) = 2$

3. The function f is defined for positive integers by

$$f(mn) = \begin{cases} f(m)f(n) & \text{if mn is divisible by 5} \\ mn & \text{otherwise} \end{cases}$$

Given that $f(25) - f(10) = f(8)$ find the value of $f(5)$

A 2
B 4
C $\sqrt{18}$
D 5
E 8

what do we know about the terms in the equation?
$f(25) = f(5) \times f(5)$
$f(10) = f(2) \times f(5) = 2 \times f(5)$
$f(8) = 8$

Let $a = f(5)$ and substitute into equation
$a^2 - 2a = 8$
$a^2 - 2a - 8 = 0$
$(a-4)(a+2) = 0$
$a = 4, -2$ $\quad f(5) = 4$

4. The function $f(n)$ is defined for positive integers n by

$$f(1) = 5 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} 3f(n) + 1 & \text{if } f(n) \text{ is even} \\ \frac{1}{2}(f(n) + 1) & \text{if } f(n) \text{ is odd} \end{cases}$$

What is the value of $f(49)$

Find the first few terms to look for any pattern

A 4

B 5

C 7

D 9

E 13

$f(1) = 5$

$f(2) = 3$

$f(3) = 2$

$f(4) = 7$

$f(5) = 4$

... keep going...

$f(6) = 13$

$f(7) = 7$

this is $f(4)$ so pattern repeats from this point

7, 4, 13, 7, 4, 13 ...

period = 3

$f(6) = f(48) = 13$

$f(49) = 7$

5. The function $f(n)$ is defined for positive integers n by

$$f(1) = 2 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} f(n) + 5 & \text{if } f(n) \text{ is even} \\ f(n) - 3 & \text{if } f(n) \text{ is odd} \end{cases}$$

Find $\sum_{r=1}^{40} f(r)$

Find the first few terms to look for any pattern

$f(1) = 2$

$f(2) = 7$

$f(3) = 4$

$f(4) = 9$

$f(5) = 6$

Notice terms will continue to alternate odd + even

odd terms form sequence 2, 4, 6 ... 40

Even terms form sequence 7, 9, 11 45

Sum both arithmetic sequences $S_n = \frac{n}{2}(a + l)$

$$\sum_{r=1}^{40} = \frac{20}{2}(2+40) + \frac{20}{2}(7+45) = 420 + 520 = 940$$

A 720

B 840

C 860

D 940

E 960

6. The function $f(n)$ is defined for positive integers n by

$$f(1) = 17 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} 2f(n) + 2 & \text{if } f(n) \text{ is even} \\ \frac{1}{2}(f(n) + 1) & \text{if } f(n) \text{ is odd} \end{cases}$$

For how many integers n in the interval $1 \le n \le 100$ is $f(n)$ odd?

Find the first few terms to look for any pattern

A 3

B 4

C 33

D 50

E 66

$f(1) = 17$

$f(2) = 9$

$f(3) = 5$

$f(4) = 3$

4 odd values

$f(5) = 2$

$f(6) = 6$

$f(7) = 14$ notice all remaining terms will be even

7. It is given that the graph of $y = f(x)$ is such that $-1 \le f(x) \le 1$ for all real x.

Find the difference between the maximum and minimum values of $(f(x))^2 - f(x)$

A $\frac{1}{4}$ B $\frac{1}{2}$ C 2 D $\frac{9}{4}$ E $\frac{9}{2}$

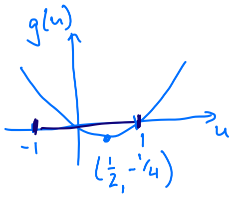
Consider $u = f(x)$

$g(u) = u^2 - u \quad -1 \le u \le 1$

Max $g(u)$ at $u = -1$ $\quad g(-1) = 1 - -1 = 2$

Min $g(u)$ at $u = \frac{1}{2}$ $\quad g(\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$

Difference $= 2 - -\frac{1}{4} = \frac{9}{4}$



8. The function $f(x)$ is such that $2f(x) + 3f(-x) \equiv 5 + (2 - x^2)\int_{-1}^{1} f(t)\ dt$

Determine the value of $\int_{-1}^{1} f(x)\ dx$

Note $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(t)\,dt = \text{constant} = k$

A 0 B 1 C 2 D 5 E 6

$2f(x) + 3f(-x) \equiv 5 + 2k - kx^2$

Integrate both sides $\int_{-1}^{1} dx$

Note $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx$

$\int_{-1}^{1} 2f(x) + 3f(-x)\,dx = \int_{-1}^{1} 5 + 2k - kx^2\,dx$

$5k = \left[5x + 2kx - \frac{1}{3}kx^3\right]_{-1}^{1}$

$5k = 5 + 2k - \frac{1}{3}k - \left(-5 - 2k + \frac{1}{3}k\right) = 10 + \frac{10}{3}k$

$\frac{5}{3}k = 10$

$k = 6$

9. The function $f(x)$ is odd and satisfies $3f(x) - 2f(-x) \equiv x^3 + \int_{-1}^{1} t^2 f(t)\ dt$

Find the value of $f(2)$

A $\frac{8}{5}$ B $\frac{26}{15}$ C 6 D $\frac{26}{3}$ E 8

For odd function $f(x) = -f(-x)$ $\quad \int_{-1}^{1} f(x)\,dx = 0$

t^2 is an even function so $t^2 f(t)$ is also odd and $\int_{-1}^{1} t^2 f(t) = 0$

$5f(x) = x^3$

$f(x) = \frac{1}{5}x^3 \qquad f(2) = \frac{8}{5}$

10. The function f is defined such that $f(x) - f(x-1) = 2$

$f(x) = f(x-1) + 2$

Find the value of $f(100) - f(11)$

What does this mean?
Value of $f(x)$ increases by 2 when x increases by 1

From $11 \to 100$ x increases by 89
So $f(x)$ increases by 178

A 89

B 90

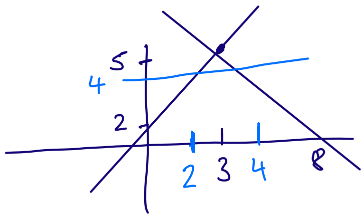
C 178

D 180

E this can not be determined

11. The function f is defined by $f(x) = \begin{cases} x+2, & x < 3 \\ 8-x, & x \geq 3 \end{cases}$

Sketch the function



Find the sum of all the real solutions to $f(f(x)) = 4$

Let $(u = f(x))$

First solve $f(u) = 4$

Both pieces have a solution

$u + 2 = 4$ $\quad$ $8 - u = 4$

$u = 2$ $(f(x) = 2)$ $\quad$ $u = 4$ $(f(x) = 4)$

Now solve

$f(x) = 2$ $\quad$ $f(x) = 4$

$x + 2 = 2$ $\quad$ $8 - x = 2$ $\quad$ As above $x = 2, 4$

$x = 0$ $\quad$ $x = 6$ $\quad$ Sum $= 0 + 6 + 2 + 4 = 12$

A 4

B 6

C 8

D 12

E 16

12. The function f is defined such that $f(xy) = f(x) \times f(y)$

Given that $f(4) = 9$ find the sum of the possible values of $f\left(\frac{1}{2}\right)$

A 0 $\quad$ B $\frac{1}{9}$ $\quad$ C $\frac{1}{3}$ $\quad$ D $\frac{9}{8}$ $\quad$ E $\sqrt{3}$

Try some values: $f(4) = f(1 \times 4) = f(1) \times f(4)$ $\quad$ $= f(2 \times 2) = f(2) \times f(2) = 9$

$f(4) = 9$ so $f(1) = 1$ $\quad$ so $f(2) = \pm 3$

$f(1) = f\left(2 \times \frac{1}{2}\right) = f(2) \times f\left(\frac{1}{2}\right)$ $\quad$ $f\left(\frac{1}{2}\right) = \frac{f(1)}{f(2)} = \frac{1}{\pm 3}$

Sum $= 0$

13. The function f is defined for the set of positive integers such that $f(xy) = f(x) + f(y)$

Given that $f(10) = 23$ and $f(40) = 37$ find $f(500)$

A 60 B 62 C 65 D 66 E 69

$f(40) = f(10 \times 4)$
$37 = 23 + f(4)$
$f(4) = 14$
$f(4) = f(2 \times 2) = 2f(2)$
$f(2) = 7$

$f(10) = f(2) + f(5)$
$23 = 7 + f(5)$
$f(5) = 16$

$f(500) = f(4 \times 125) = f(4) + 3f(5)$
$= 14 + 3(16)$
$= 62$

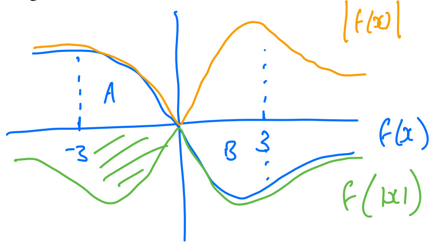
14. The function $f(x)$ is such that $f(0) = 0$ and $xf(x) < 0$ for $x \neq 0$

You are given that ① $\int_{-3}^{3} f(x)\ dx = -2$ and ② $\int_{-3}^{3} |f(x)|\ dx = 14$

Find $\int_{-3}^{0} f(|x|)\ dx$

A -12
B -8
C -6
D 6
E 8
F 12

Sketch a random continuous function to represent $f(x)$ in 2nd + 4th quadrants



Let A, B represent areas

so ① $A - B = -2$

② $A + B = 14$

Solve to find $A = 6$, $B = 8$

Need to find // $= -B = -8$

Area is positive
Integral is below
x-axis so negative

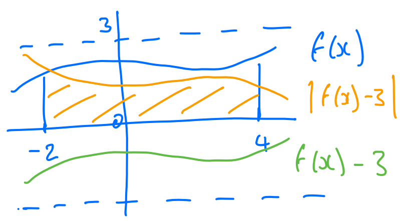
15. The function $f(x)$ is such that $0 \leq f(x) \leq 3$ for all real x.

You are given that $\int_{-2}^{4} \left|f(x) - 3\right|\ dx = 10$ ①

Find $\int_{-2}^{4} f(x)\ dx$

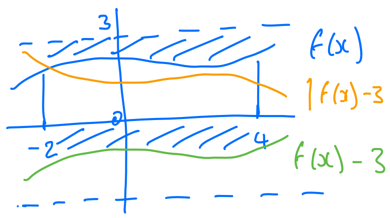
A -28 B -13 C 8 D 13 E 28

Sketch $f(x)$ and
$|f(x) - 3|$



From ① // $= 10$

This is the same
area as //



$\int_{-2}^{4} f(x)\ dx =$ Area of rectangle $- 10$

$= 6 \times 3 - 10 = 8$