This worksheet looks at ways of solving questions involving the modulus function $f(x) = |x|$.

This function always gives the **absolute value** or **magnitude**, which means if the expression inside the vertical lines is positive or zero, it stays the same, but if it is negative it becomes positive.

Some examples: $|5| = 5$ $|-3| = 3$ $|0| = 0$

In other words if $x \geq 0$ $|x| = x$ and if $x < 0$ $|x| = -x$

so if $|x| = 4$ then either $x = 4$, or $x = -4$

Further examples: $f(x) = |2x - 6|$ Solve $2x - 6 \geq 0$ to determine $x \geq 3$

Therefore $f(x) = \begin{cases} -2x + 6 & x < 3 \\ 2x - 6 & x \geq 3 \end{cases}$

$g(x) = x|2 - x|$ Solve $2 - x \geq 0$ to determine $x \leq 2$

Therefore $g(x) = \begin{cases} 2x - x^2 & x \leq 2 \\ x^2 - 2x & x > 2 \end{cases}$

Note that $|a - b|$ can also be thought of as the distance between a and b. Try this out with various positive and negative values for a and b. This means that $|x - 4| = 3$ can be solved by thinking about which values are a distance of 3 away from 4 (solutions are 1 and 7).

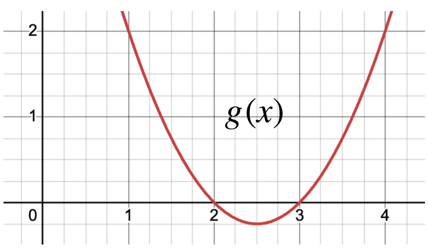
In order to sketch a graph of a modulus function you should consider for which values of x the expression inside the vertical lines is positive / negative.

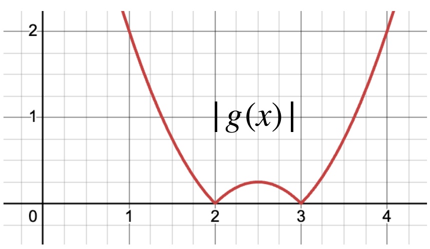
Consider $f(x) = |g(x)| = |x^2 - 5x + 6| = |(x - 2)(x - 3)|$

Now $x^2 - 5x + 6 \geq 0$ when $x \leq 2$ or $x \geq 3$ so for these values $f(x) = x^2 - 5x + 6 = g(x)$

and $x^2 - 5x + 6 < 0$ when $2 < x < 3$ so for these values $f(x) = -x^2 + 5x - 6 = -g(x)$

This means we can sketch the graph of $|g(x)|$ by sketching $g(x)$ first for all values of x, and then reflecting anything below the x-axis, in the x-axis in order to achieve $-g(x)$ for these values of x.





It is generally helpful to do a quick sketch of a modulus function if you are solving an equation or inequality in order to assess where the solutions lie and if you need to consider $g(x)$ or $-g(x)$. You can also often identify the number of solutions from a sketch.

Considering where $g(x)$ is positive / negative can also help us to remove the modulus signs to achieve a more manageable function for example when we need to integrate. We can split the integral into two or more parts, integrating $g(x)$ between limits where $g(x) \geq 0$ and $-g(x)$ between limits where $g(x) < 0$.

For the previous example:

$$\int_0^6 |x^2 - 5x + 6| \ dx = \int_0^2 x^2 - 5x + 6 \ dx + \int_2^3 -x^2 + 5x - 6 \ dx + \int_3^6 x^2 - 5x + 6 \ dx$$

Finally consider how you would sketch the graph of $f(|x|)$ given $f(x)$

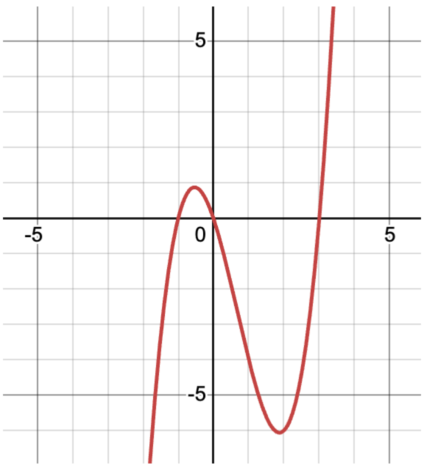
If $x \geq 0$ then $|x| = x$ and so $f(|x|) = f(x)$ (the graph looks the same as $f(x)$ for $x \geq 0$)

If $x < 0$ then $|x| = -x$ and so $f(|x|) = f(-x)$ (this is a reflection of $f(x)$ in the y-axis).

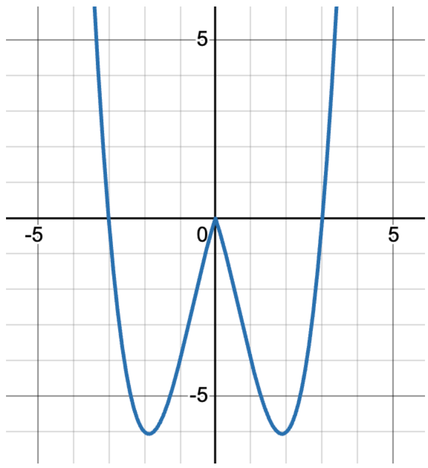
So we need to draw $f(x)$ for $x \geq 0$ and $f(-x)$ for $x < 0$, which means:

first sketch $f(x)$ for $x \geq 0$ and then reflect it in the y-axis.

Example $f(x) = x(x-3)(x+1)$

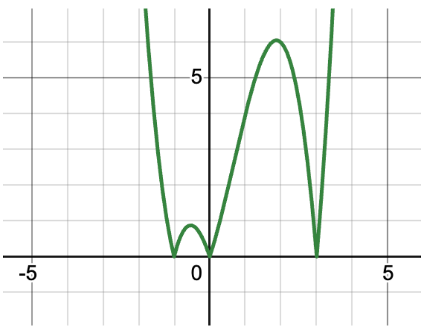


$f(|x|)$



Note that $f(|x|)$ can be negative as well as positive.

$|f(x)|$



The graph of $|f(x)|$ is shown here for comparison.

TMUA style questions - Modulus Function

1. x and y satisfy $|3-2x| \leq 7$ and $|y+3| \leq 6$

What is the greatest possible value of $|x+y|$

A 7

B 8

C 11

D 14

E There is no greatest possible value

2. Find the set of values of x for which $|x^2-4| < 4-2x$

A $-4 < x < -2$

B $-4 < x < 0$

C $-4 < x < 2$

D $-2 < x < 0$

E $0 < x < 2$

3. What is the product of the solutions of the following equation

$$4-|x| = |4x+8|$$

A $\frac{16}{15}$ B $\frac{48}{25}$ C $\frac{16}{5}$ D 4 E $\frac{16}{3}$

4. How many solutions are there to the following equation:

$$|x^3| + |x - 2| = |4x|$$

A 0

B 1

C 2

D 3

E 4

5. What is the minimum value of the function:

$$|x + 1| + |x - 1| + |x - 2| + |x - 4|$$

A 0

B 2

C 4

D 6

E 8

6. Find the area between the curves with equations $y = 2|x + 1|$ and $y = 6 - 2|x + 1|$

A $\sqrt{5}$

B 6

C $3\sqrt{5}$

D 9

E $6\sqrt{5}$

F 18

7. x and y satisfy $|y - x| \le 2$ and $y \le 4 - x^2$

What is the greatest possible value of $|xy|$

A 3 B 4 C 8 D 15 E 16

8. Consider the finite area between the curves $y = k|x| - 3$ and $y = -\frac{1}{k}|x| + 2$

What is the sum of the possible values of k that give an area of 10 square units?

A 2 B $\frac{5}{2}$ C 3 D $\frac{10}{3}$ E 5

9. Find the finite area between the coordinate axes and the curve $y = x^2 - 2|x| - 8$?

A $-\frac{160}{3}$ B $-\frac{80}{3}$ C -4 D $-\frac{16}{3}$ E 0

10. Evaluate the following integral $\int_0^3 x|6-3x|\ dx$

A 0 B 4 C 8 D 12 E 15

11. Evaluate $\int_{-2}^{1} x\left|(x+1)\right|\ dx$

A $-\frac{3}{2}$ B $-\frac{1}{3}$ C $-\frac{1}{6}$ D $\frac{1}{6}$ E $\frac{1}{3}$ F $\frac{3}{2}$

12. Evaluate the following integral $\int_0^2 2x(|1-x|)-x|x|\ dx$

A $-\frac{2}{3}$ B $-\frac{8}{3}$ C -4 D $-\frac{16}{3}$ E 0

13. Arrange the following integrals in order from smallest to largest:

P: $\displaystyle\int_{-2}^{2} \left|3+x\right| - \left|3-x\right| \ dx$

Q: $\displaystyle\int_{0}^{2} x\left|3-x\right| \ dx$

R: $\displaystyle\int_{-2}^{0} x\left|3+x\right| \ dx$

A P Q R

B P R Q

C Q P R

D Q R P

E R P Q

F R Q P

14. The function $f(x)$ is such that $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$

You are given that $\displaystyle\int_{-3}^{3} f(x) \ dx = 0$ $\displaystyle\int_{-3}^{3} f(|x|) \ dx = 6$

Find $\displaystyle\int_{-3}^{0} |f(x)| \ dx$

A -6 B -3 C 0 D 3 E 6

15. How many solutions are there to $\left|3-\left|2-|1-x|\right|\right|-1=0$

A 2 B 3 C 4 D 5 E 6

16. The equation $|x-a|+|x+a|=10-x$ where $a>0$ has exactly 1 real solution. What is the value of a ?

A 0 B $\frac{5}{3}$ C $\frac{10}{3}$ D 5 E 10

17. How many positive solutions for x are there for the equation:

$$\int_0^x \left|1-|t|\right|-1 \; dt = 0$$

A 0
B 1
C 2
D 4
E infinitely many

18. How many values of k give 3 distinct real solutions to $|x-k| = x^2 - 4x + 4$

A 0

B 1

C 2

D 3

E 4

F infinitely many

19. Consider the inequality $m|x| + 2 < x^2 + 3x + 6$ where m is a real constant.

What is the complete set of values for m that make this inequality true for all real x?

A $m < 1$

B $m < \frac{3}{2}$

C $m < 2$

D $-7 < m < 1$

E $-1 < m < 7$

20. The equation $\left| |x-1| - 3 \right| - 5 = k$ has exactly 3 solutions.

What is the value of k ?

A -2 B -1 C 1 D 3 E 5