TMUA Practice - Abstract Functions - Notes

This worksheet focuses on ways of solving questions involving a function $f(x)$ that is not explicitly defined, or only partially defined.

You may be given some information about the function that can help such as:

$xf(x) > 0$ - this means that if $x > 0$ then $f(x) > 0$, and similarly if $x < 0$ then $f(x) < 0$

A graph like this will only be in the 1st and 3rd quadrants (and usually goes through the origin).

$f(x)$ is an **even** function - this means $f(x) = f(-x)$ so there is a line of symmetry along the y-axis

Examples include $f(x) = x^2$ $f(x) = cosx$ $f(x) = \frac{1}{x^2}$

$f(x)$ is an **odd** function - this means $f(x) = -f(-x)$ so there is rotational symmetry about the origin

Examples include $f(x) = x^3$ $f(x) = sinx$ $f(x) = \frac{1}{x}$

$f(x)$ is a **self-inverse** function - this means $ff(x) = x$ so applying the function twice gets you back where you started.

Examples include $f(x) = -x$ $f(x) = \frac{1}{x}$

If an equation is given in terms of $f(x)$, it can sometimes help to substitute a few values for x to see if this produces any equations that you can work with (including simultaneous equations), or whether you can spot any repeating patterns, or cycles. Is the function periodic, or even a constant.

If a second function is described in terms of $f(x)$, eg $2f(x) - \left(f(x)\right)^3$,

it can help to make a substitution $u = f(x)$, so that you can then work with $g(u) = 2u - u^3$, and consider properties of this function. Remember the range of $f(x)$ becomes the domain for $g(u)$.

When given information about the value of integrals between particular x-coordinates, be careful to distinguish between these values which can be positive or negative, and the corresponding area it refers to which is always positive (but can be represented by the absolute value of the integral).

Draw a sketch to represent the required areas, label these and form equations using the information given (see Integration worksheet for more practice on this technique).

1. The function f is defined such that $3f(x) - f(-x) = 8x + 10$
find the value of $f(2)$

A 5

B 6

C 7

D 8

E 9

2. The function f satisfies $2f(x) + f\left(\dfrac{x+5}{x-1}\right) = 3x, \quad x \in \mathbb{R}$
Find the value of $f(3)$

A 2

B 3

C 4

D 5

E 6

3. The function f is defined for positive integers by

$$f(mn) = \begin{cases} f(m)f(n) & \text{if mn is divisible by 5} \\ mn & \text{otherwise} \end{cases}$$

Given that $f(25) - f(10) = f(8)$ find the value of $f(5)$

A 2

B 4

C $\sqrt{18}$

D 5

E 8

4. The function $f(n)$ is defined for positive integers n by

$$f(1) = 5 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} 3f(n) + 1 & \text{if } f(n) \text{ is even} \\ \frac{1}{2}(f(n) + 1) & \text{if } f(n) \text{ is odd} \end{cases}$$

What is the value of $f(49)$

A 4

B 5

C 7

D 9

E 13

5. The function $f(n)$ is defined for positive integers n by

$$f(1) = 2 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} f(n) + 5 & \text{if } f(n) \text{ is even} \\ f(n) - 3 & \text{if } f(n) \text{ is odd} \end{cases}$$

Find $\displaystyle\sum_{r=1}^{40} f(r)$

A 720

B 840

C 860

D 940

E 960

6. The function $f(n)$ is defined for positive integers n by

$$f(1) = 17 \qquad \text{and for } n > 1, \quad f(n+1) = \begin{cases} 2f(n) + 2 & \text{if } f(n) \text{ is even} \\ \frac{1}{2}(f(n) + 1) & \text{if } f(n) \text{ is odd} \end{cases}$$

For how many integers n in the interval $1 \leq n \leq 100$ is $f(n)$ odd?

A 3

B 4

C 33

D 50

E 66

7. It is given that the graph of $y = f(x)$ is such that $-1 \leq f(x) \leq 1$ for all real x.

Find the difference between the maximum and minimum values of $\left(f(x)\right)^2 - f(x)$

A $\frac{1}{4}$ B $\frac{1}{2}$ C 2 D $\frac{9}{4}$ E $\frac{9}{2}$

8. The function $f(x)$ is such that $2f(x) + 3f(-x) \equiv 5 + (2 - x^2)\int_{-1}^{1} f(t)\ dt$

Determine the value of $\int_{-1}^{1} f(x)\ dx$

A 0 B 1 C 2 D 5 E 6

9. The function $f(x)$ is odd and satisfies $3f(x) - 2f(-x) \equiv x^3 + \int_{-1}^{1} t^2 f(t)\ dt$

Find the value of $f(2)$

A $\frac{8}{5}$ B $\frac{26}{15}$ C 6 D $\frac{26}{3}$ E 8

10. The function f is defined such that $f(x) - f(x-1) = 2$

Find the value of $f(100) - f(11)$

A 89

B 90

C 178

D 180

E this can not be determined

11. The function f is defined by $f(x) = \begin{cases} x+2, & x < 3 \\ 8-x, & x \geq 3 \end{cases}$

Find the sum of all the real solutions to $f(f(x)) = 4$

A 4

B 6

C 8

D 12

E 16

12. The function f is defined such that $f(xy) = f(x) \times f(y)$

Given that $f(4) = 9$ find the sum of the possible values of $f\left(\frac{1}{2}\right)$

A 0 B $\frac{1}{9}$ C $\frac{1}{3}$ D $\frac{9}{8}$ E $\sqrt{3}$

13. The function f is defined for the set of positive integers such that $f(xy) = f(x) + f(y)$

Given that $f(10) = 23$ and $f(40) = 37$ find $f(500)$

A 60 B 62 C 65 D 66 E 69

14. The function $f(x)$ is such that $f(0) = 0$ and $xf(x) < 0$ for $x \neq 0$

You are given that $\int_{-3}^{3} f(x)\ dx = -2$ and $\int_{-3}^{3} |f(x)|\ dx = 14$

Find $\int_{-3}^{0} f(|x|)\ dx$

A -12
B -8
C -6
D 6
E 8
F 12

15. The function $f(x)$ is such that $0 \leq f(x) \leq 3$ for all real x.

You are given that $\int_{-2}^{4} \left| f(x) - 3 \right|\ dx = 10$

Find $\int_{-2}^{4} f(x)\ dx$

A -28 B -13 C 8 D 13 E 28