

1. Differentiate the following expressions with respect to x

$$\text{a) } \frac{2x + 4x^2}{\sqrt{x}} = 2x^{1/2} + 4x^{3/2}$$

$$\frac{d}{dx} = x^{-1/2} + 6x^{1/2} = \frac{1}{\sqrt{x}} + 6\sqrt{x}$$

$$\text{b) } \frac{1 - \sqrt{x}}{4x^3} = \frac{1}{4}x^{-3} - \frac{1}{4}x^{-5/2}$$

$$\frac{d}{dx} = -\frac{3}{4}x^{-4} + \frac{5}{8}x^{-7/2} = -\frac{3}{4x^4} + \frac{5}{8x^{7/2}}$$

$$\text{c) } 2\sqrt{x}\left(\frac{5}{x} + x^2\right) = 10x^{-1/2} + 2x^{5/2}$$

$$\frac{d}{dx} = -5x^{-3/2} + 5x^{3/2} = -\frac{5}{x^{3/2}} + 5x^{3/2}$$

$$\text{d) } \frac{(3 + 2\sqrt{x})^2}{4x} = \frac{9 + 12x^{1/2} + 4x}{4x} = \frac{9}{4}x^{-1} + 3x^{-1/2} + 1$$

$$\frac{d}{dx} = -\frac{9}{4}x^{-2} - \frac{3}{2}x^{-3/2} = -\frac{9}{4x^2} - \frac{3}{2x^{3/2}}$$

$$\text{e) } \frac{(2x - 1)(x^2 + 4)}{2\sqrt[3]{x}} = \frac{2x^3 - x^2 + 8x - 4}{2x^{1/3}} = x^{8/3} - \frac{1}{2}x^{5/3} + 4x^{2/3} - 2x^{-1/3}$$

$$\frac{d}{dx} = \frac{8}{3}x^{5/3} - \frac{5}{6}x^{2/3} + \frac{8}{3}x^{-1/3} + \frac{2}{3}x^{-4/3}$$

2. Find the equation of the tangent to the curve at the point given

a) $y = 2\sqrt{x} - \frac{6}{\sqrt{x}}$ where $x = 4$ $y = 1$

$$y = 2x^{1/2} - 6x^{-1/2}$$

$$\frac{dy}{dx} = x^{-1/2} + 3x^{-3/2}$$

$$= \frac{1}{\sqrt{x}} + \frac{3}{\sqrt{x^3}}$$

At $x = 4$ $\frac{dy}{dx} = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$

$$y - 1 = \frac{7}{8}(x - 4)$$

$$8y - 8 = 7x - 28$$

$$8y - 7x + 20 = 0$$

b) $y = 3x^{3/2} - \frac{32}{x}$ where $x = 4$ $y = 16$

$$\frac{dy}{dx} = \frac{9}{2}x^{1/2} + 32x^{-2}$$

$$\left. \frac{dy}{dx} \right|_4 = 9 + 2 = 11$$

$$y - 16 = 11(x - 4)$$

$$y - 16 = 11x - 44$$

$$y = 11x - 28$$

Find the equation of the normal to the curve at the point given

c) $y = x^2(x - 6) + \frac{5}{x} - 1$ where $x = 1$ $y = -5 + 5 - 1 = -1$

$$= x^3 - 6x^2 + 5x^{-1} - 1$$

$$y' = 3x^2 - 12x - 5x^{-2}$$

At $x = 1$ $y' = 3 - 12 - 5 = -14$

$$m = \frac{1}{14}$$

$$y + 1 = \frac{1}{14}(x - 1)$$

$$14y + 14 = x - 1$$

$$14y - x + 15 = 0$$

d) $y = 2x^2 - 4x^{3/2} - \frac{8}{x} - 1$ where $x = 4$ $y = 32 - 32 - 2 - 1 = -3$

$$y' = 4x - 6x^{1/2} + 8x^{-2}$$

$$y'(4) = 16 - 12 + \frac{1}{2} = \frac{9}{2}$$

$$m = -\frac{2}{9}$$

$$y + 3 = -\frac{2}{9}(x - 4)$$

$$9y + 27 = -2x + 8$$

$$9y + 2x + 19 = 0$$

e) The tangent to the curve $y = x^3 - x$ at the point P (1,0) meets the curve again at the point Q.

What is the distance PQ?

$$y' = 3x^2 - 1$$

$$y'(1) = 2$$

$$y = 2(x - 1)$$

$$y = 2x - 2$$

At P, Q

$$x^3 - x = 2x - 2$$

$$x^3 - 3x + 2 = 0$$

$$(x - 1)(x^2 + x - 2) = 0$$

$$(x - 1)^2(x + 2) = 0$$

At Q $x = -2$ $y = -6$ (-2, -6)

$$PQ = \sqrt{3^2 + 6^2}$$

$$= \sqrt{45}$$

$$= 3\sqrt{5}$$

f) The normal to the curve $y = (x - 1)(x^2 + 4x + 5)$ at the point where $x = -1$ meets the coordinate axes at the points P and Q.

What is the area of triangle OPQ, where O is the origin?

$$y = x^3 + 3x^2 + x - 5$$

$$y' = 3x^2 + 6x + 1$$

$$y'(-1) = 3 - 6 + 1 = -2$$

$$m = \frac{1}{2}$$

$$y + 4 = \frac{1}{2}(x + 1)$$

$$x = 0 \quad y + 4 = \frac{1}{2}$$

$$y = -\frac{7}{2}$$

$$y = 0 \quad 4 = \frac{1}{2}(x + 1)$$

$$x = 7$$

P, Q (0, - $\frac{7}{2}$), (7, 0)

$$\text{Area OPQ} = \frac{1}{2} \times \frac{7}{2} \times 7$$

$$= \frac{49}{4}$$

3. Find the coordinates of the stationary point(s) of the following equations, and determine if they are maximums, minimums, or points of inflexion.

a) $y = x^3 - 3x^2 - 9x + 3$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$x = 3$ $y = 27 - 27 - 27 + 3 = -24$

$x = -1$ $y = -1 - 3 + 9 + 3 = 8$

$\frac{d^2y}{dx^2} = 6x - 6$

$\frac{d^2y}{dx^2} \Big|_{x=3} = 18 - 6 > 0$ so minimum

$\frac{d^2y}{dx^2} \Big|_{x=-1} = -6 - 6 < 0$ so maximum

$(3, -24)$ min
 $(-1, 8)$ max

b) $y = x^2 + \frac{16}{x}$

$$y' = 2x - 16x^{-2} = 0$$

$$y'' = 2 + 32x^{-3}$$

$$2x = \frac{16}{x^2}$$

$$x^3 = 8$$

$$x = 2$$

$$y = 4 + 8 = 12$$

$y'' = 2 + 4 > 0$
so $(2, 12)$ is minimum

c) $y = 3x^4 + 16x^3 + 24x^2 + 3$

$$y' = 12x^3 + 48x^2 + 48x = 0$$

$$x(x^2 + 4x + 4) = 0$$

$$x(x+2)^2 = 0$$

$$y'' = 36x^2 + 96x + 48$$

$x = 0$ $y = 3$

$x = -2$ $y = 3(16) - 8(16) + (24)4 + 3 = -80 + 99 = 19$

At $x = 0$ $y'' > 0$ minimum $(0, 3)$

$x = -2$ $y'' = 36(4) - 96(2) + 48 = 12(12 - 16 + 4) = 0$

Consider gradient function $f'(x) = x(x+2)^2$ repeated root at -2

$f(x)$ has point of inflexion at $(-2, 19)$

4 Find the range of values of x , for which y is a decreasing function

a) $y = x^3 - 3x^2 - 9x + 10$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 < 0$$

$$x^2 - 2x - 3 < 0$$

$$(x-3)(x+1) < 0$$

$-1 < x < 3$

b) $y = -x^3 + 7x^2 - 8x - 10$

$$\frac{dy}{dx} = -3x^2 + 14x - 8 < 0$$

$$3x^2 - 14x + 8 > 0$$

$$(3x-2)(x-4) > 0$$

$x < \frac{2}{3}, x > 4$

c) $y = \frac{x^2 + 4}{4x} \quad x \neq 0$

$$y = \frac{x}{4} + \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{4} - \frac{1}{x^2} < 0$$

$$\frac{1}{x^2} > \frac{1}{4}$$

$x^2 < 4$
 $-2 < x < 2$

d) $y = \sqrt[3]{x} + \frac{27}{x}$

$$= x^{1/3} + 27x^{-1}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} - \frac{27}{x^2} < 0$$

$$\frac{1}{3}x^{4/3} - 27 < 0$$

$$x^{4/3} < 81$$

$$x^4 < 81^3$$

$x^4 - 3^{12} < 0$
 $(x^2 - 3^6)(x^2 + 3^6) < 0$

$x^2 - 3^6 < 0$
 $(x - 3^3)(x + 3^3) < 0$
 $-27 < x < 27$