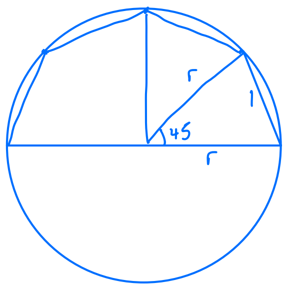
TMUA style questions - Geometry

1. A regular octagon with side length 1 is drawn inside a circle so that each vertex of the octagon lies on the perimeter of the circle.

   What is the area of the circle?

   A $\dfrac{(2-\sqrt{2})\pi}{2}$ B $\dfrac{(1+\sqrt{2})\pi}{2}$ C $\sqrt{2}\pi$ D $\dfrac{(2+\sqrt{2})\pi}{2}$ E 2π



Cosine rule: $\cos 45 = \dfrac{r^2+r^2-1}{2r^2}$

$\dfrac{\sqrt{2}}{2} = \dfrac{2r^2-1}{2r^2}$

$\sqrt{2}r^2 = 2r^2-1$

$r^2(2-\sqrt{2}) = 1$

$r^2 = \dfrac{1}{2-\sqrt{2}} \times \dfrac{2+\sqrt{2}}{2+\sqrt{2}}$

$= \dfrac{2+\sqrt{2}}{4-2} = \dfrac{2+\sqrt{2}}{2}$

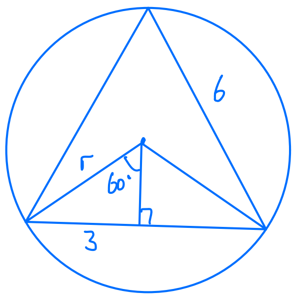
Area of circle $= \pi r^2$

$= \left(\dfrac{2+\sqrt{2}}{2}\right)\pi$

2. An equilateral triangle with side length 6cm is inscribed in a circle (the circle passes through each of the vertices of the triangle).

   Which of the following is an expression for the circumference of the circle, in cm.

   A 6π B $4\pi\sqrt{3}$ C $\dfrac{10\pi\sqrt{3}}{3}$ D $3\pi\sqrt{3}$ E $\dfrac{4\pi\sqrt{3}}{3}$



$\sin 60 = \dfrac{3}{r}$ $r = \dfrac{3}{\sin 60} = 3 \times \dfrac{2}{\sqrt{3}} = 2\sqrt{3}$

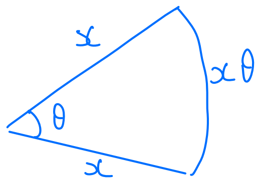
$C = 2\pi r = 4\pi\sqrt{3}$

3. A sector of a circle of radius x cm contains an angle of θ radians.

The perimeter of the sector is 12 cm.

What is the maximum possible value of the area of the sector as x varies?

A 3 B 6 C 9 D 12 E 18



$P = 2x + x\theta = 12$

$\theta = \dfrac{12 - 2x}{x}$

$A = \frac{1}{2}x^2\left(\frac{12-2x}{x}\right) = 6x - x^2$

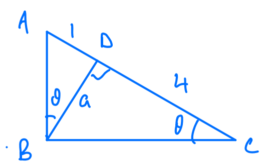
$\frac{dA}{dx} = 6 - 2x = 0 \quad x = 3$

$A = \frac{1}{2}(9)(2) = 9$

4. Triangle ABC is right angled at B. The perpendicular from B to the hypotenuse meets AC at D. Given that the ratio of AD:CD is 1:4, which fraction best represents the following

$$\frac{\text{length of hypotenuse AC}}{\text{sum of lengths of AB and BC}} \ ?$$

A $\dfrac{2\sqrt{3}}{5}$ B $\dfrac{\sqrt{5}}{3}$ C $\dfrac{4}{5}$ D $\dfrac{\sqrt{3}}{2}$ E $\dfrac{2\sqrt{5}}{5}$



As we are only interested in the ratio of lengths choose $AD = 1$ $DC = 4$ $AC = 5$

Similar triangles ABD : BCD : $\frac{1}{a} = \frac{a}{4}$ $a^2 = 4$ $a = 2$

Pythagoras : $AB^2 = 1^2 + 2^2$ $AB = \sqrt{5}$
$BC^2 = 2^2 + 4^2$ $BC = \sqrt{20} = 2\sqrt{5}$

fraction $= \dfrac{5}{3\sqrt{5}} = \dfrac{\sqrt{5}}{3}$

5. Given that $x^2 + y^2 = 1$, what is the greatest possible value of $x - 3y$ $= k$

Find max k

$y = \frac{x}{3} - \frac{k}{3}$

$(0, -\frac{k}{3})$ $(k, 0)$

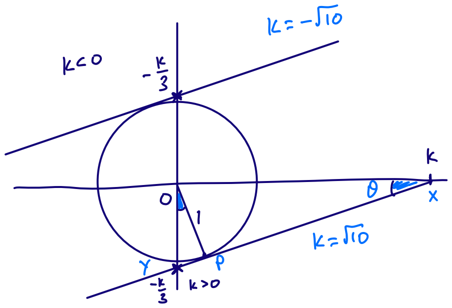
A $3\sqrt{2}$

B $\sqrt{10}$

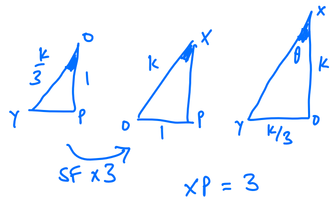
C 3

D $2\sqrt{2}$

E 1



Sketch similar triangles



$xP = 3$

$YP = 1/3$

Pythagoras OPX $1 + 3^2 = k^2$

$k = \pm\sqrt{10}$

max $k = +\sqrt{10}$

OR Use trigonometry

OYP : $\cos\theta = \frac{3}{k}$ OPX : $\sin\theta = \frac{1}{k}$

$\frac{9}{k^2} + \frac{1}{k^2} = 1$

$k^2 = 10$

$k = \pm\sqrt{10}$

6. Circle C_1 has equation $(x+3)^2 + (y-3)^2 = 5$ Centre $(-3, 3)$ $r = \sqrt{5}$

Circle C_2 has equation $(x-9)^2 + (y-3)^2 = 20$ Centre $(9, 3)$ $r = 2\sqrt{5}$

The straight line L passes between these circles with a positive gradient and is a tangent to both circles. The angle between L and the x-axis is θ. Find $\cos\theta$

A $\frac{\sqrt{5}}{6}$ B $\frac{\sqrt{5}}{4}$ C $\frac{2}{3}$ D $\frac{\sqrt{11}}{4}$ E $\frac{\sqrt{3}}{2}$

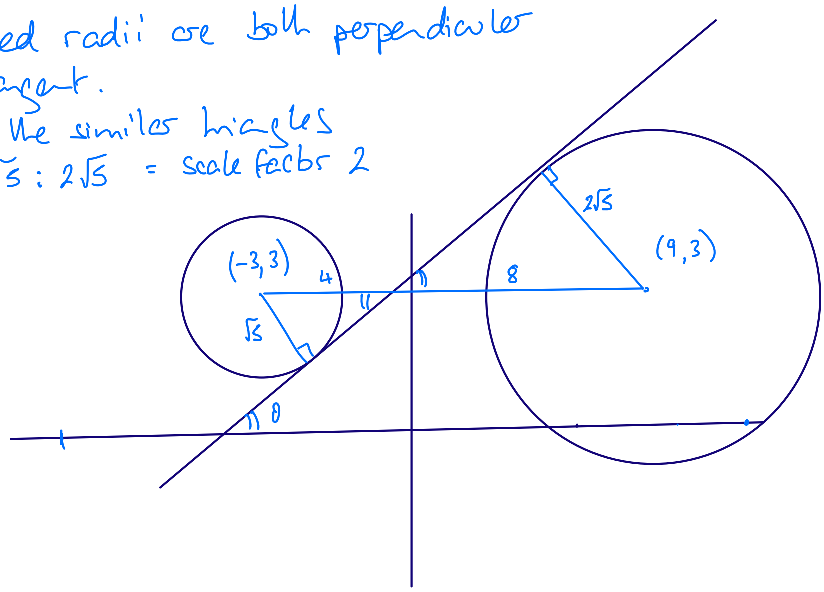
Marked radii are both perpendicular to tangent.

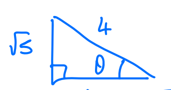
Spot the similar triangles

$\sqrt{5} : 2\sqrt{5}$ = scale factor 2

Distance between centres = 12

in ratio $1:2 = 4:8$





$\sqrt{16-5} = \sqrt{11}$

$\cos\theta = \frac{\sqrt{11}}{4}$

7. A triangle ABC is to be drawn with the following measurements.

$AB = 8$ cm, angle $BAC = 30°$, $BC = x$ cm.

For which values of x can a unique triangle ABC be drawn ?

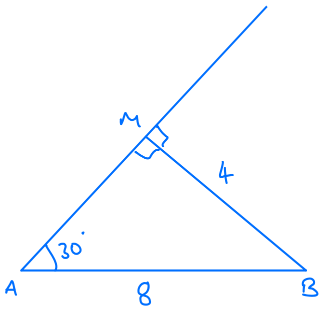
A $x = 4$ or $x \geq 8$

B $4 \leq x \leq 8$

C $x \geq 4$

D $x \geq 8$

E $0 \leq x \leq 4$ or $x = 8$



$MB = 8 \sin 30$
$= 4$

2 triangles
$4 < BC < 8$

1 triangle
$BC = 4$
$BC \geq 8$

0 triangles
$BC < 4$

8. A triangle ABC is to be drawn with the following measurements.

$AC = b$ $BC = 2a$ angle $ABC = 45°$ where a and b are constants

For which value of b can two distinct triangles ABC be drawn ?

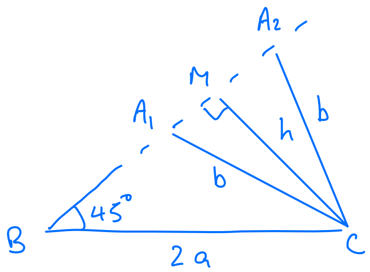
A $b = \dfrac{\sqrt{2}}{2}\, a$

B $b = a$

C $b = \sqrt{2}\, a$

D $b = \dfrac{3}{2}\, a$

E $b = 2a$



MBC : $\sin 45 = \dfrac{h}{2a}$

$h = \sqrt{2}\, a$

For 2 triangles need :
$h < b < BC$

$\sqrt{2}a < b < 2a$

only value that fits
is $\dfrac{3}{2}a$

9. A triangle ABC is to be drawn with the following measurements.

$AB = x \qquad BC = 6 \qquad \text{angle } BAC = 60° \qquad \text{angle } ABC > 90°$

Find the range of values for x such that only one possible triangle can be drawn, satisfying the conditions above.

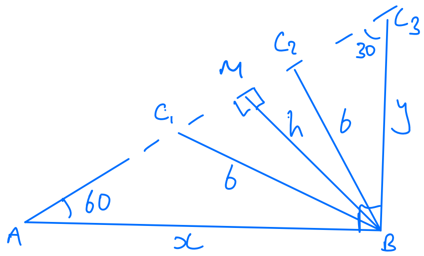
A $\quad 0 < x < 2\sqrt{3}$

B $\quad 2\sqrt{3} < x < 6$

C $\quad 2\sqrt{3} < x \le 4\sqrt{3}$

D $\quad 0 < x < 6$

E $\quad 6 < x \le 4\sqrt{3}$



Draw in C_3 such that ABC_3 is 90°.
Need $BC_3 < 6$ for ABC obtuse

ABC_3 $\quad \tan 60 = \frac{y}{x}$

$y = \sqrt{3}x$

$\therefore \sqrt{3}x < 6$

$x < 2\sqrt{3}$

ABM $\quad \sin 60 = \frac{h}{x} \quad h = \frac{\sqrt{3}}{2}x$

for unique triangle
either $h = 6$ or $6 > x$

has ABC acute

so $x < 6$

combined range

$0 < x < 2\sqrt{3}$

10. A triangle ABC is to be drawn with the following measurements.

$AB = \sqrt{13}\,a \qquad BC = \sqrt{10}\,a \qquad \text{angle } BAC = \theta° \qquad \text{where } \tan\theta = \frac{3}{2}$

Given that two possible triangles can be drawn, what is ratio of the area of the larger triangle to the area of the smaller triangle?

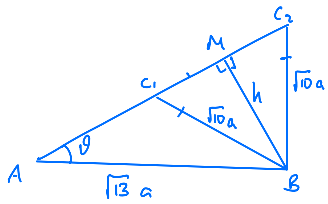
A $\quad \sqrt{13} : 3$

B $\quad \sqrt{13} : \sqrt{10}$

C $\quad 3:1$

D $\quad \sqrt{10} : 1$

E $\quad 13 : 3$



$\tan\theta = \frac{h}{AM} = \frac{3}{2}$

$AM = \frac{2}{3}h$

AMB : $AM^2 + MB^2 = AB^2$

$\frac{4}{9}h^2 + h^2 = 13a^2$

$\frac{13}{9}h^2 = 13a^2$

$h = 3a$

$AM = 2a$

C_1MB : $C_1M^2 + h^2 = 10a^2$

$C_1M^2 = a^2$

$C_1M = a \quad = C_2M$

$AC_1 = a \qquad AC_2 = 3a$

$AC_2 : AC_1 = 3 : 1 \quad =$ ratio of areas*

*Proof Area $AC_1B = \frac{1}{2} AB\, AC_1 \sin\theta$

Area $AC_2B = \frac{1}{2} AB\, AC_2 \sin\theta$

Ratio of areas $= \frac{1}{2} AB\, AC_2 \sin\theta : \frac{1}{2} AB\, AC_1 \sin\theta$

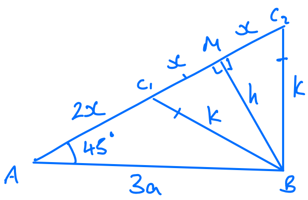
$AC_2 : AC_1$

11. Consider a triangle ABC with the following measurements.

$AB = 3a \qquad BC = k \qquad \text{angle } BAC = 45°$

Given that two possible triangles can be drawn, find the value of k that means the area of the larger triangle is double the area of the smaller one. $\Rightarrow AC_2 : AC_1 = 2:1$

A $2a$ B $\sqrt{5}\,a$ C $3a$ D $3\sqrt{2}\,a$ E $6a$



$\sin 45 = \frac{h}{3a}$

$h = \frac{3a\sqrt{2}}{2}$

Pythagoras $9x^2 + h^2 = 9a^2$

$9x^2 = 9a^2 - \frac{9}{2}a^2 = \frac{9}{2}a^2$

$x^2 = \frac{1}{2}a^2$

Can also get this from cosine rule

$k^2 = 4x^2 + 9a^2 - 12ax\frac{\sqrt{2}}{2}$ $\qquad k^2 = 16x^2 + 9a^2 - 24ax\frac{\sqrt{2}}{2}$

$4x^2 - 6\sqrt{2}ax = 16x^2 - 12\sqrt{2}ax$

$12x^2 - 6\sqrt{2}ax = 0$

$6x(2x - \sqrt{2}a) = 0$

$x = \frac{\sqrt{2}}{2}a$

Pythagoras $k^2 = x^2 + h^2$

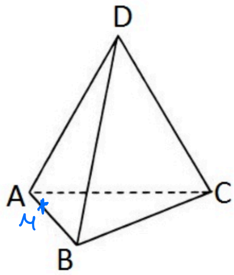
$= \frac{1}{2}a^2 + \frac{9}{2}a^2 = 5a^2$

$k = \sqrt{5}a$

12. The diagram shows a tetrahedron with base ABC and vertex D. All the edges of the tetrahedron are the same length. The base ABC is stuck to a table.

M is the point that lies on AB such that the ratio of AM:BM is 1:3.

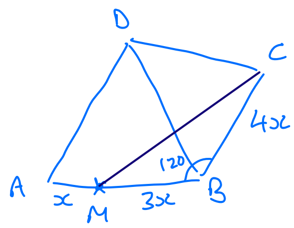
Find the ratio of MC:BC where MC is the shortest distance from M to C along the **upper** outer surfaces of the tetrahedron (not across the base).



A $\sqrt{3} : 2$ B $\sqrt{13} : 4$ C $\sqrt{13} : 3$ D $\sqrt{37} : 4$ E $\sqrt{21} : 4$

Identify the possible paths from M to C

1

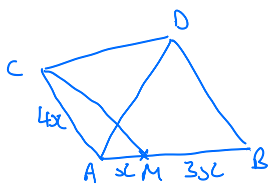


$MC^2 = 9x^2 + 16x^2 - 24\left(-\frac{1}{2}\right)x^2$

$MC^2 = 25x^2 + 12x^2$

$MC = \sqrt{37}\,x$

2



$MC^2 = 16x^2 + x^2 - 8\left(-\frac{1}{2}\right)x^2$

$MC^2 = 21x^2$

$MC = \sqrt{21}\,x$ shortest

$\sqrt{21}x : 4x$

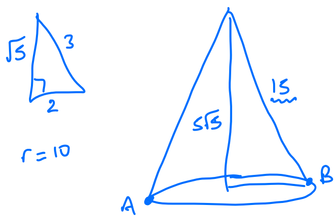
13. A right cone has height $5\sqrt{5}$ cm and slant height 15 cm.

Points A and B lie on the circumference of the base and are diametrically opposite.

What is the shortest distance along the curved surface from A to B?

A $15\sqrt{3}$ B $5\sqrt{3}\pi$ C 30 D 10π E $5\sqrt{5}\pi$

Work out the radius using this similar triangle



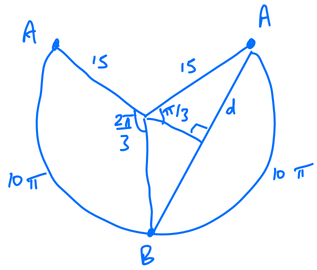
$r = 10$

$C = 20\pi$

Curved surface of cone opens to form sector

Arc length $= 20\pi$

Angle of sector $= \frac{20\pi}{15} = \frac{4\pi}{3}$



$\sin\frac{\pi}{3} = \frac{d}{15}$

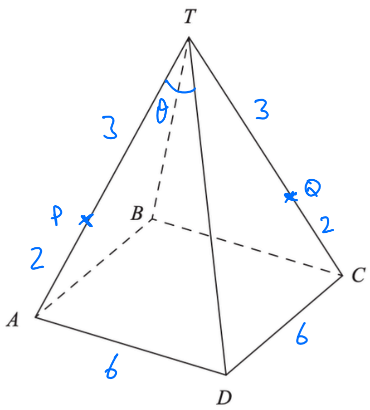
$d = \frac{15\sqrt{3}}{2}$

$AB = 15\sqrt{3}$

14. A square based pyramid has base ABCD and vertex T. The base edges have length 6m, and the sloping edges (AT, BT, CT, DT) have length 5m.

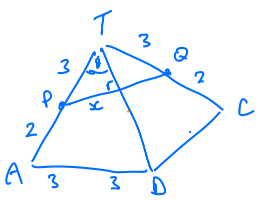
Point P is on AT such that PT is 3m.
Point Q is on CT such that QT is 3m.

Find the shortest distance, in metres, along the outer surface of the pyramid from P to Q.



A $\frac{36}{5}$ B 6 C $\frac{35}{6}$ D $\frac{144}{25}$ E $\frac{27}{5}$

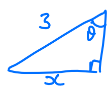
2D net



Cosine rule ATD

$\cos\theta = \frac{25 + 25 - 36}{50} = \frac{7}{25}$

$\sin\theta = \frac{24}{25}$



$\sin\theta = \frac{x}{3} = \frac{24}{25}$

$x = \frac{72}{25}$

$PQ = \frac{144}{25}$